Berkeley Math Circle: Monthly Contest 7

Due April 10, 2018

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 7 is due on April 10, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 7, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 7

1. The numbers 1 through 7 are written on a blackboard. Each minute, two numbers are erased and their sum is written instead. Find all possible values for the final number left on the board.
2. Show that for any positive integer n, the numbers $3n+2$ and $4n+3$ have no common factors greater than 1.
3. A *partition* of a positive integer n is a way of writing n as an unordered sum of not necessarily distinct positive integer parts. Show that the number of partitions of n with all odd parts equals the number of partitions with all distinct parts.
4. We have 2009 prime numbers $p_1 < p_2 < p_3 < \cdots < p_{2009}$ such that $p_1^2+p_2^2+\cdots+p_{2009}^2$ is a perfect square. Prove that p_1 divides $p_{2009}^2 - p_{2008}^2$.

5. Two numbers are *relatively prime* if their only common divisor is 1. For a positive integer n, let $\phi(n)$ be the number of positive integers less than or equal to n and relatively prime to n. Write $d \mid n$ if d is a divisor of n. Find

$$\sum_{d \mid n} \phi(d).$$

In other words, compute the sum of $\phi(d)$ across all divisors of n.

6. Define the function $s\colon \mathbb{Z}^2 \to \mathbb{Z}$ by

$$s(n,k) = \begin{cases} 1 & n \le k \\ -1 & n > k. \end{cases}$$

Prove that if integers $x_1, \ldots, x_{100}$ satisfy $x_i^2 = 1$ for each i, then

$$\prod_{n=1}^{100} \left(\sum_{k=1}^{100} s(n,k) x_k \right) = 0.$$

7. Let Ω be a fixed circle and $\overline{BC}$ a fixed chord of that circle which is not a diameter. A variable diameter $\overline{AD}$ of Ω, with A on minor arc $\widehat{BC}$, is chosen. Line BD meets line AC at E, while line CD meets line AB at F. Points P and Q are the reflections of D over B and C.

   (a) Prove that points A, P, F, E, Q lie on a circle, say Γ.

   (b) The tangents to Γ at E and F meet at P. Prove that line AP passes through a fixed point as A varies.