# Berkeley Math Circle: Monthly Contest 6

Due March 13, 2018

## Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 6 is due on March 13, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 6, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

## Problems for Contest 6

1. Ten fair coins are flipped. Given that there are at least nine heads, what is the probability that all the coins show heads?

2. Is there a positive integer $n$ for which $n(n+1)$ is a perfect square?

3. Prove that for any positive integer $n$, we have

$$\prod_{k=1}^{n} \operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n}{k} \right\rfloor\right) = n!.$$

4. Let $ABC$ be a triangle and let $P$ be a point inside it satisfying $\angle ABP = \angle PCA$. Let $Q$ be the reflection of $P$ across the midpoint of $\overline{BC}$. Prove that $\angle BAP = \angle CAQ$.

5. Find the smallest prime $p > 100$ for which there exists an integer $a > 1$ such that $p$ divides $\frac{a^{89}-1}{a-1}$.

6. Convex quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle $\Omega$; the diagonals of $ABCD$ meet at $X$. Suppose $AD < AB$, the circumcircle of triangle $BCX$ intersects segment $AB$ at a point $Y \neq B$, and ray $\overrightarrow{CY}$ meets $\Omega$ again at a point $Z \neq C$. Prove that ray $\overrightarrow{DY}$ bisects angle $ZDB$.

7. Prove that there are infinitely many positive integers $n$ for which $n^2 + 1$ has no repeated prime factors (that is, $n^2 + 1$ is *squarefree*).