# Berkeley Math Circle: Monthly Contest 5

Due February 13, 2018

## Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 5 is due on February 13, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

## Problems for Contest 5

1. How many ways are there to color the vertices of an equilateral triangle red, green, or blue, where colorings that can be obtained from each other by rotation or reflection are considered the same?

2. Let $\left\{ {n \atop k} \right\}$ be the number of ways to partition $n$ objects into $k$ nonempty subsets. For example, $\left\{ {4 \atop 2} \right\} = 7$ since the four objects $A$, $B$, $C$, $D$ can be partitioned in exactly seven ways: $\{A, BCD\}$, $\{B, CDA\}$, $\{C, DAB\}$, $\{D, ABC\}$, $\{AB, CD\}$, $\{AC, BD\}$, $\{AD, BC\}$.

   Prove that
   $$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

3. Show that every positive integer $n$ can be uniquely written in the form
   $$n = \sum_{i=0}^{k} d_i \cdot 2^i = d_0 + 2d_1 + \dots 2^k d_k$$
   for some positive integer $k$, where each $d_i$ either 1 or 2.

4. A group of prisoners play the following game: each is given a red or black hat, and they stand in a line such that each can see the hats of everyone in front of him, but not his own or those of anyone behind him. Starting from the end of the line, each in turn must guess his hat color, and they will win the game and all be freed if at most one of them guess incorrectly.

   They are allowed to discuss a strategy beforehand, and they can hear the previous guesses, but no other communication is allowed during the game. Show that they can always win.

5. Let $ABC$ be an equilateral triangle. Points $L$, $P$, and $Q$ lie on the segments $AB$, $AC$, and $BC$, respectively, and are such that $PCQL$ is a parallelogram. Let $M$ be the midpoint of $AB$. The circle with center $M$ passing through $C$ intersects the circle with diameter $CL$ again at $T$. Prove that lines $AQ$, $BP$, and $LT$ are concurrent.

6. Suppose 4951 points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.

7. For positive real numbers $a$, $b$, $c$ prove that

$$\frac{1+bc}{a} + \frac{1+ca}{b} + \frac{1+ab}{c} > \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$