Berkeley Math Circle: Monthly Contest 3

Due December 5, 2018

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 3 is due on December 5, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 3, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 3

1. How many ways are there to place three rooks on an 8×8 chessboard such that the rooks are in different columns and different rows?

2. Prove that if a, b, c, d are positive integers then $ab + bc + cd + da$ is not prime.

3. Alice and Bob play a game. There are 9 cards numbered 1 through 9 on a table, and the players alternate taking the cards, with Alice going first. A player wins if at any point they hold three cards with sum 15; if all nine cards are taken before this occurs, the game is a tie. Does either player have a winning strategy?

4. Let ABC be a triangle and P a point inside it. Lines AP, BP, CP meet the opposite sides at D, E, F. Assume that the three quadrilaterals $PDCE$, $PEAF$, $PFBD$ are all bicentric. Prove that triangle ABC is equilateral.

   (A quadrilateral is bicentric if it can be inscribed inside a circle, and it also can have a circle inscribed inside it.)

5. For which positive integers n does the polynomial $P(X) = X^n + X^{n-1} + \cdots + 1$ have a real root?

6. Let a and b be positive integers, and let A and B be finite disjoint sets of positive integers. Assume that for every $i \in A \cup B$, we have $i + a \in A$ or $i - b \in B$. Prove that $a|A| = b|B|$.

7. Find all ordered triples of non-negative integers (a, b, c) such that a^2+2b+c, b^2+2c+a, and $c^2 + 2a + b$ are all perfect squares.