Berkeley Math Circle: Monthly Contest 2

Due October 31, 2018

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 2 is due on October 31, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 2

1. You are out walking and see a group of rhinoceroses (which each have two horns and four legs) and triceratopses (which each have three horns and four legs). If you count 31 horns and 48 legs, how many triceratopses are there?

2. Let n be a positive integer. There are n islands, and you want to build some number of bridges so that there is a path between any two islands. (Each bridge joins a pair of islands, and can be traveled in both directions.) At least how many bridges must you build?

3. Let n be a nonnegative integer. Prove that the numbers $n+2$ and n^2+n+1 cannot both be perfect cubes.

4. The endpoints of a chord ST with constant length are moving along a semicircle with diameter AB. Let M be the midpoint of ST and P the foot of the perpendicular from S to AB. Prove that the angle SPM is independent of the location of ST.

5. Let n be a positive integer. Is it possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the arithmetic mean of any two of these numbers is not equal to some number between them?

6. Let a, b, x, y, z be positive real numbers. Prove the inequality

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

7. Let $a_0 = a_1 = 1$ and $a_{n+1} = 7a_n - a_{n-1} - 2$ for all positive integers n. Prove that a_n is a perfect square for all n.