Berkeley Math Circle: Monthly Contest 1

Due October 3, 2018

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 1 is due on October 3, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 1

1. Do there exist positive irrational numbers x and y such that $x + y$ and xy are both rational? If so, give an example; if not, explain why not.

2. Let n be an odd positive integer not divisible by 3. Show that $n^2 - 1$ is divisible by 24.

3. Four cars A, B, C, and D travel at constant speeds on the same road (not necessarily in the same direction). Car A passed B and C at 8am and 9am, respectively, and met D at 10am. Car D met B and C at 12pm and 2pm, respectively. Determine at what time B passed C. (The times given are within a single day.)

4. A row of fifty coins with integer denominations is given, such that the sum of the denominations is odd. Alice and Bob alternate taking either coin at the left end of the row or the right end of the row, with Alice playing first. Prove that Alice can always ensure she gets more than half the money.

5. Let a, b, c be positive real numbers such that $abc = 1$. Simplify

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca}.$$

6. Two triangles ABC and XYZ have a common circumcircle. Suppose the nine-point circle γ of $\triangle ABC$ passes through the midpoints of $\overline{XY}$ and $\overline{XZ}$. Prove that γ also passes through the midpoint of $\overline{YZ}$.

7. Let G be a simple graph with k connected components, which have $a_1, \ldots, a_k$ vertices, respectively. Determine the number of ways to add $k-1$ edges to G to form a connected graph, in terms of the numbers a_i.