Berkeley Math Circle: Monthly Contest 7

Due April 11, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 7 is due on April 11, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 7, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 7

1. Lisa considers the number

$$x = \frac{1}{1^1} + \frac{1}{2^2} + \cdots + \frac{1}{100^{100}}.$$

   Lisa wants to know what x is when rounded to the nearest integer. Help her determine its value.

2. A number is called *cool* if it is the sum of two nonnegative perfect squares. For example the numbers 17 and 25 are cool because $17 = 4^2 + 1^2$ and $25 = 5^2 + 0^2$, but the number 15 is not cool.

   Show that if k is cool, then $2k$ is cool.

3. Victoria paints every positive integer either pink or blue. Is it possible that both conditions below are satisfied?
   - For every positive integer n, the numbers n and $n + 5$ are different colors.
   - For every positive integer n, the numbers n and $2n$ are different colors.

4. Let H be the orthocenter of an acute triangle ABC. The circumcircle ω of triangle HAB intersects line BC at the point $D \neq B$. Let P be the intersection of the line DH and the line segment AC, and let Q be the circumcenter of triangle ADP. Show that the center of ω lies on the circumcircle of triangle BDQ.

5. Prove that

$$64\frac{abcd+1}{(a+b+c+d)^2} \leq a^2+b^2+c^2+d^2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}$$

for $a, b, c, d > 0$.

6. Let $a_1, a_2, \ldots, a_{1000}$ be real numbers such that

$$\begin{aligned} a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots + a_{1000} \cdot 1000 &= 0 \\ a_1 \cdot 1^2 + a_2 \cdot 2^2 + a_3 \cdot 3^2 + \cdots + a_{1000} \cdot 1000^2 &= 0 \\ a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_{1000} \cdot 1000^3 &= 0 \\ &\vdots \\ a_1 \cdot 1^{999} + a_2 \cdot 2^{999} + a_3 \cdot 3^{999} + \cdots + a_{1000} \cdot 1000^{999} &= 0 \\ a_1 \cdot 1^{1000} + a_2 \cdot 2^{1000} + a_3 \cdot 3^{1000} + \cdots + a_{1000} \cdot 1000^{1000} &= 1. \end{aligned}$$

What is the value of a_1?

7. Determine all primes p such that there exists an integer x satisfying $x^{2010} + x^{2009} + \cdots + 1 \equiv p^{2010} \pmod{p^{2011}}$.