Berkeley Math Circle: Monthly Contest 5

Due February 14, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 5 is due on February 14, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 5, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 5

1. A bird thinks the number $2n^2 + 29$ is prime for every positive integer n. Find a counterexample to the bird's conjecture.

2. An iguana writes the number 1 on the blackboard. Every minute afterwards, if the number x is written, the iguana erases it and either writes $\frac{1}{x}$ or $x + 1$. Can the iguana eventually write the number $\frac{20}{17}$?

3. We define a *chessboard polygon* to be a polygon whose edges are situated along lines of the form $x = a$ and $y = b$, where a and b are integers. These lines divide the interior into unit squares, which we call cells.

   Let n and k be positive integers. Assume that a square can be partitioned into n congruent chessboard polygons of k cells each. Prove that this square may also be partitioned into k congruent chessboard polygons of n cells each.

4. Let ABC be a triangle, I the incenter, and D the intersection of lines AI and BC. The perpendicular bisector of AD meets BI and CI at P and Q. Show that I is the orthocenter of triangle PQD.

5. Each of the positive integers a_1, a_2, $\ldots$, a_n is less than 2016, and the least common multiple of any two is greater than 2016. Show that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 1 + \frac{n}{2016}.$$

6. Let a_1, a_2, $\ldots$ be an infinite sequence of positive real numbers which satisfies

$$a_{n+1} \geq a_n^2 + \frac{1}{5}$$

for every positive integer n. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}$ for each positive integer n.

7. Prove that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{m+1}{n} + \frac{n+1}{m}$$

is an integer.