Berkeley Math Circle: Monthly Contest 4

Due January 17, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 4 is due on January 17, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 4, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 4

1. On an 6×6 chessboard, we randomly place counters on three different squares. What is the probability that no two counters are in the same row or column?

2. Alice picks an *odd* integer n and writes the fraction
$$\frac{2n+2}{3n+2}.$$
Show that this fraction is already in lowest terms. (For example, if $n = 5$ this is the fraction $\frac{12}{17}$.)

3. Let ABC be a triangle. A line is drawn not passing through any vertex of ABC. Prove that some side of ABC is not cut by the line.

4. A sequence $a_1, a_2, \ldots$ of positive integers satisfies
$$a_{n+1} = a_n^3 + 103$$
for every positive integer n. Prove that the sequence contains at most one perfect square.

5. Show that n divides $\varphi(a^n - 1)$ for any integers a and n, where φ is Euler's totient function.

6. Let a, b, c be pairwise distinct integers. Prove that

$$\frac{a^3 + b^3 + c^3}{3} \geq abc + \sqrt{3(ab + bc + ca + 1)}.$$

7. Let $AXYZB$ be a convex pentagon inscribed in a semicircle with diameter $\overline{AB}$, and let K be the foot of the altitude from Y to $\overline{AB}$. Let O denote the midpoint of $\overline{AB}$ and L the intersection of $\overline{XZ}$ with $\overline{YO}$. Select a point M on line KL with $MA = MB$, and finally, let I be the reflection of O across $\overline{XZ}$. Prove that if quadrilateral $XKOZ$ is cyclic then so is quadrilateral $YOMI$.