Berkeley Math Circle: Monthly Contest 2

Due November 7, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 2 is due on November 7, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 2

1. A rectangle $\mathcal{R}$ has perimeter is $2\sqrt{2018}$ and the diagonal has length $\sqrt{2017}$. What is its area?
2. For odd positive integers a, b, c prove that
$$a^4 + b^4 + 2017 \neq c^4.$$
3. In quadrilateral $ABCD$ we have $AB = 7$, $BC = 24$, $CD = 15$, $DA = 20$, and $AC = 25$. What is the length of BD?
4. A country has 50 states. How many ways are there to join some pairs of them by two-way flights such that every state has an odd number of flights departing it?
5. Determine whether there exist polynomials $A(x)$, $B(x)$, $P(y)$, $Q(y)$ with real coefficients satisfying
$$x + y + (xy)^{2017} = A(x)P(y) + B(x)Q(y).$$

6. Solve the equation $a^2 + b^2 + c^2 = (ab)^2$ over the integers.

7. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a cyclic octagon. Let B_i by the intersection of A_iA_{i+1} and $A_{i+3}A_{i+4}$ (where indices are taken modulo 8). Prove that $B_1, B_2, \ldots, B_8$ lie on a conic.