Berkeley Math Circle: Monthly Contest 1

Due October 3, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 1 is due on October 3, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 1

1. Find the number of divisors of $2^9 \cdot 3^{14}$.

2. Find all ordered triples (a, b, c) of positive integers with $a^2 + b^2 = 4c + 3$.

3. In the game *Kayles*, there is a line of bowling pins, and two players take turns knocking over one pin or two adjacent pins. The player who makes the last move (by knocking over the last pin) wins.

   Show that the first player can always win no matter what the second player does.

   (Two pins are *adjacent* if they are next to each other in the original lineup. Two pins do *not* become adjacent if the pins between them are knocked over.)

4. In $\triangle ABC$, points D and E lie on side BC and AC respectively such that $AD \perp BC$ and $DE \perp AC$. The circumcircle of $\triangle ABD$ meets segment BE at point F (other than B). Ray AF meets segment DE at point P. Prove that $DP/PE = CD/DB$.

5. Show that for positive real numbers a, b, and c,

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)}.$$

6. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(f(x)+xf(y)) = x + f(x)y,$$

where $\mathbb{Q}$ is the set of rational numbers.

7. Evaluate the sum

$$\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\frac{P_i-1}{P_{i+1}}\right) = \frac{1}{3}+\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{4}{7}+\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{4}{7}\cdot\frac{6}{11}+\dots,$$

where P_n denotes the n^{th} prime number.