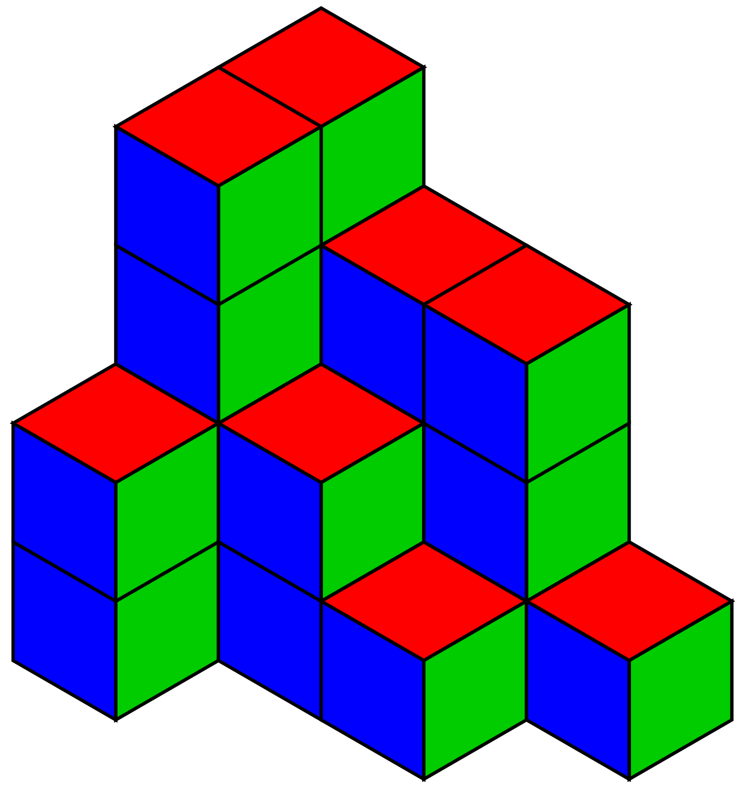
4	3	3	1
4	2	1	0
2	0	0	0



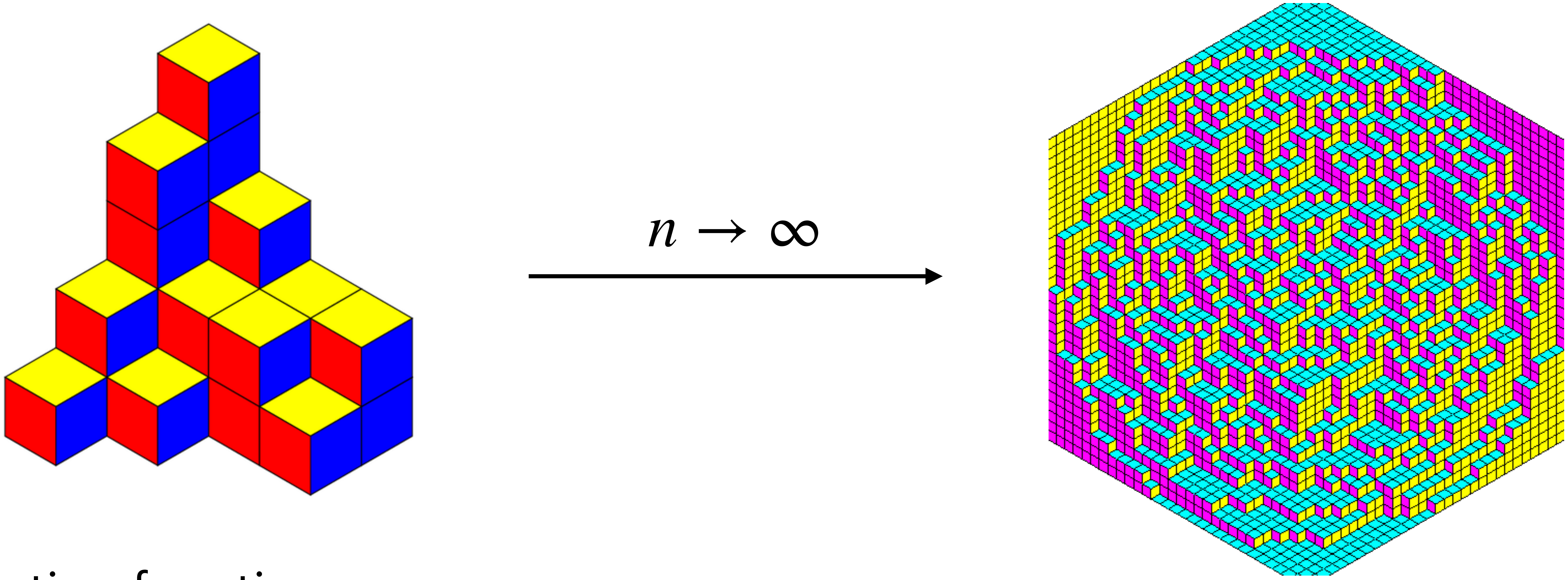
Plane Partitions

or how to stack blocks

Peter Koroteev

Lecture at Stanford Math Circle 3/4/2025

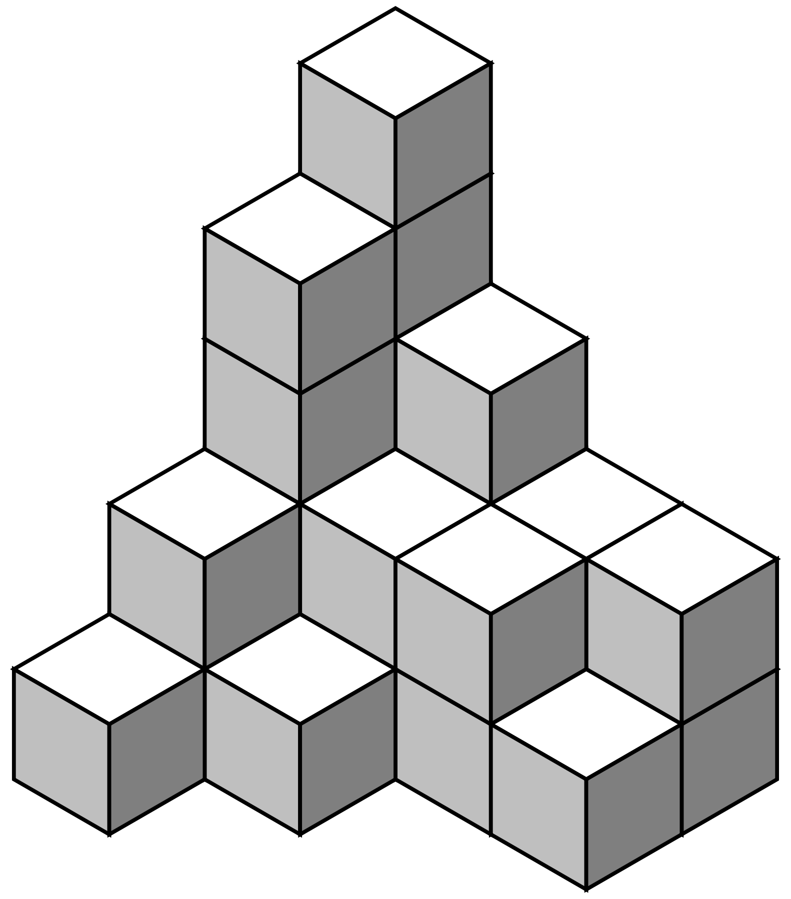
Plane (3d) Partitions



Generating function

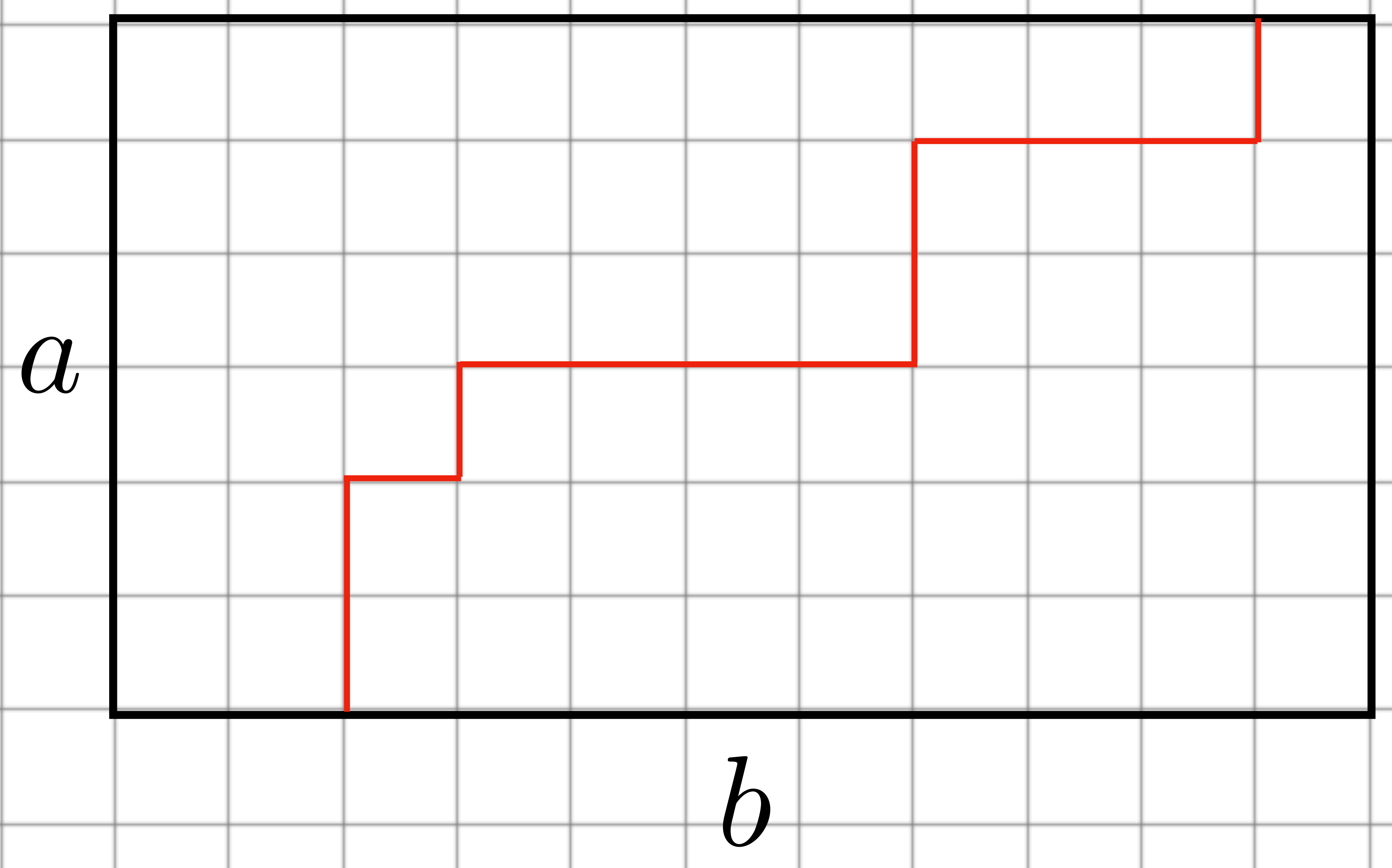
$$\sum_{n=0}^{\infty} \mathrm{PL}(n) x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + \cdots$$

A View from Above



5	3	2	2
4	2	2	1
2	1		
1			

Paths in a Box



What is the number of paths from the bottom-left corner to the top-right corner?

A bijection between staircase paths and integer partitions

(q-)Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$n! = n(n-1)\cdots 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

$$[n] = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

$$[n]! = [n][n-1]\ldots[1]$$

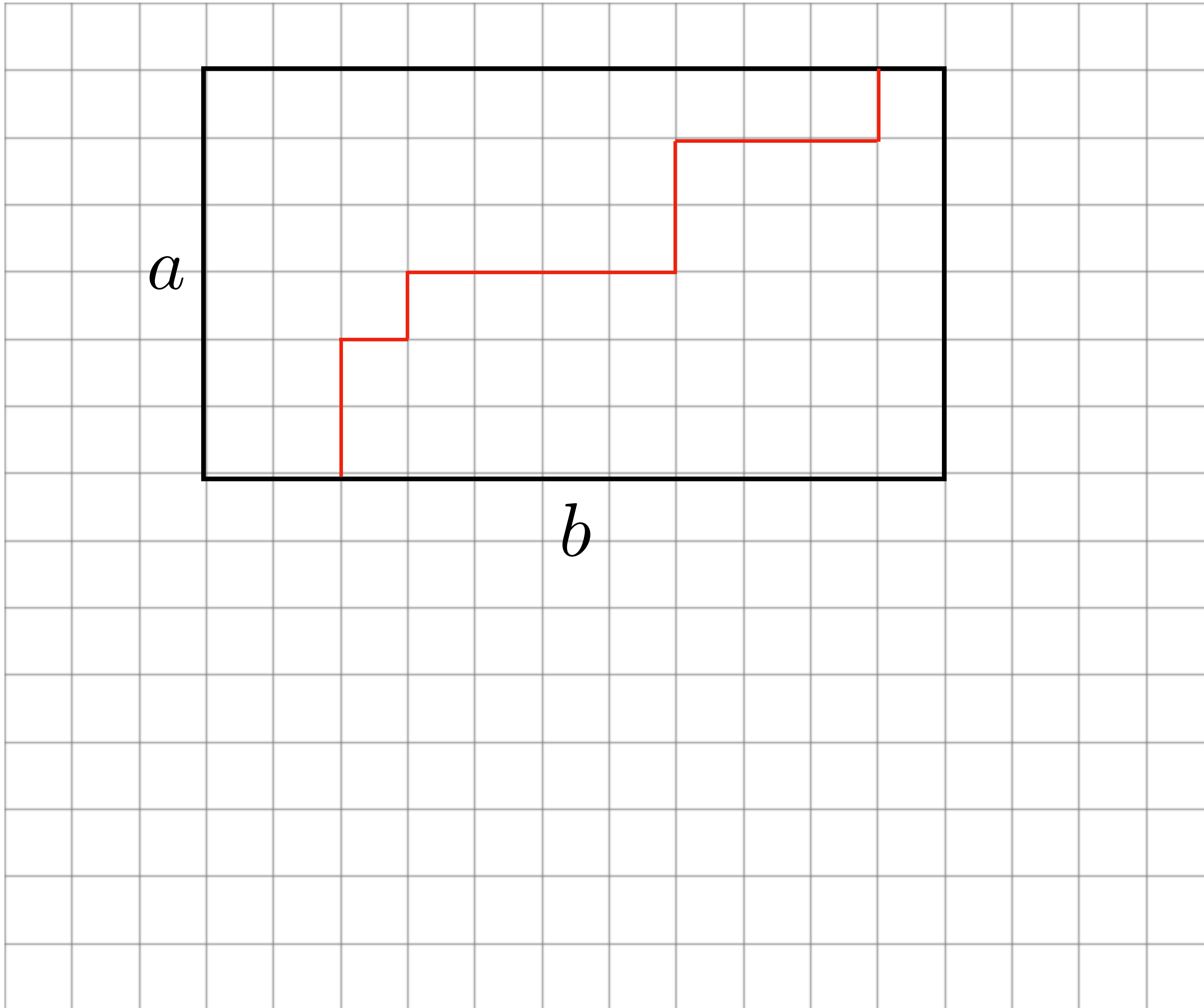
What happens with $[n]$ as $q \to 1$?

Box Counting Formula

Theorem

$$\sum_{\lambda} q^{|\lambda|} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

$|\lambda|$ — size of partition (# boxes)



Lemma

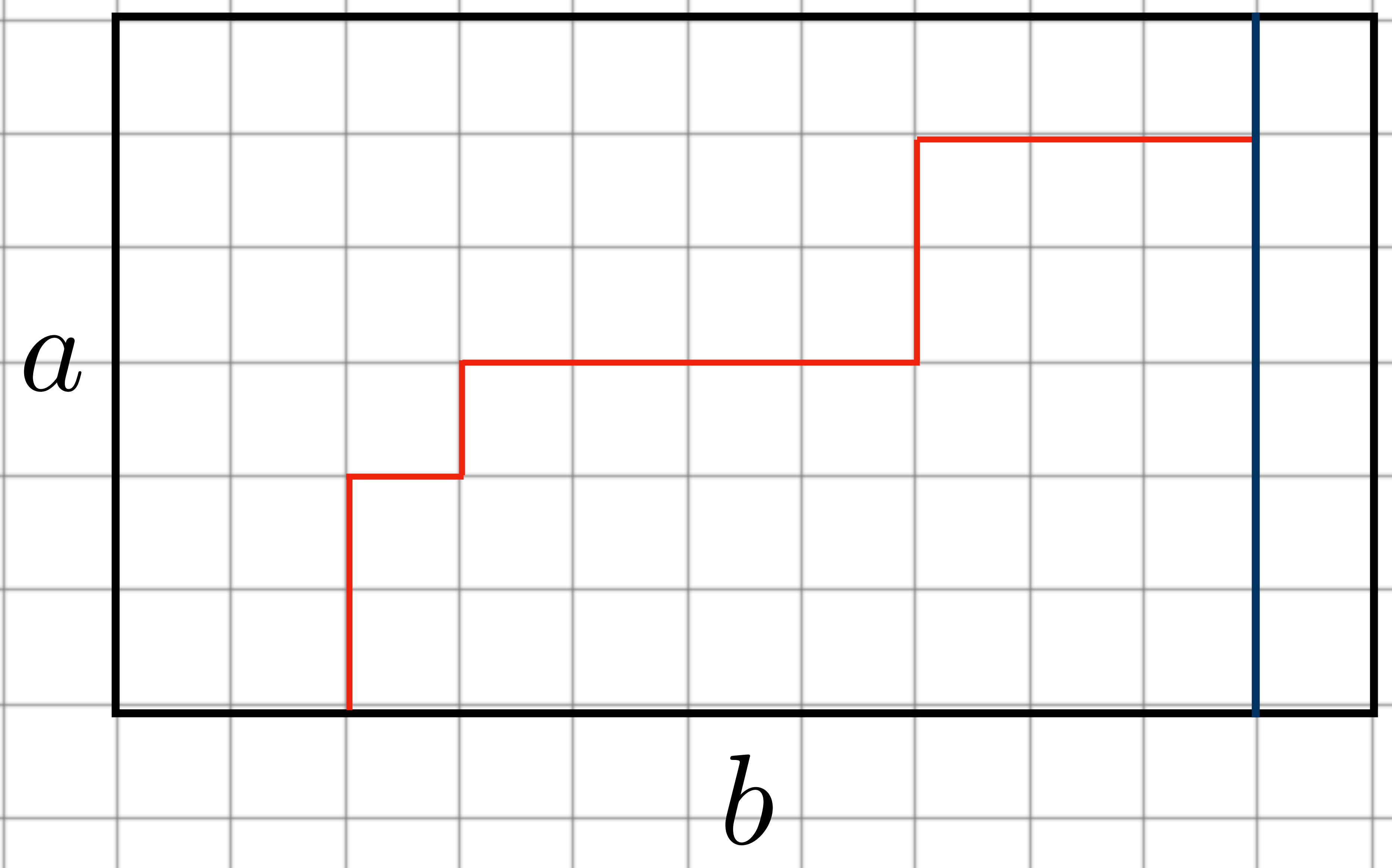
$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

It is a q-deformation of the Pascal's triangle recurrence relation

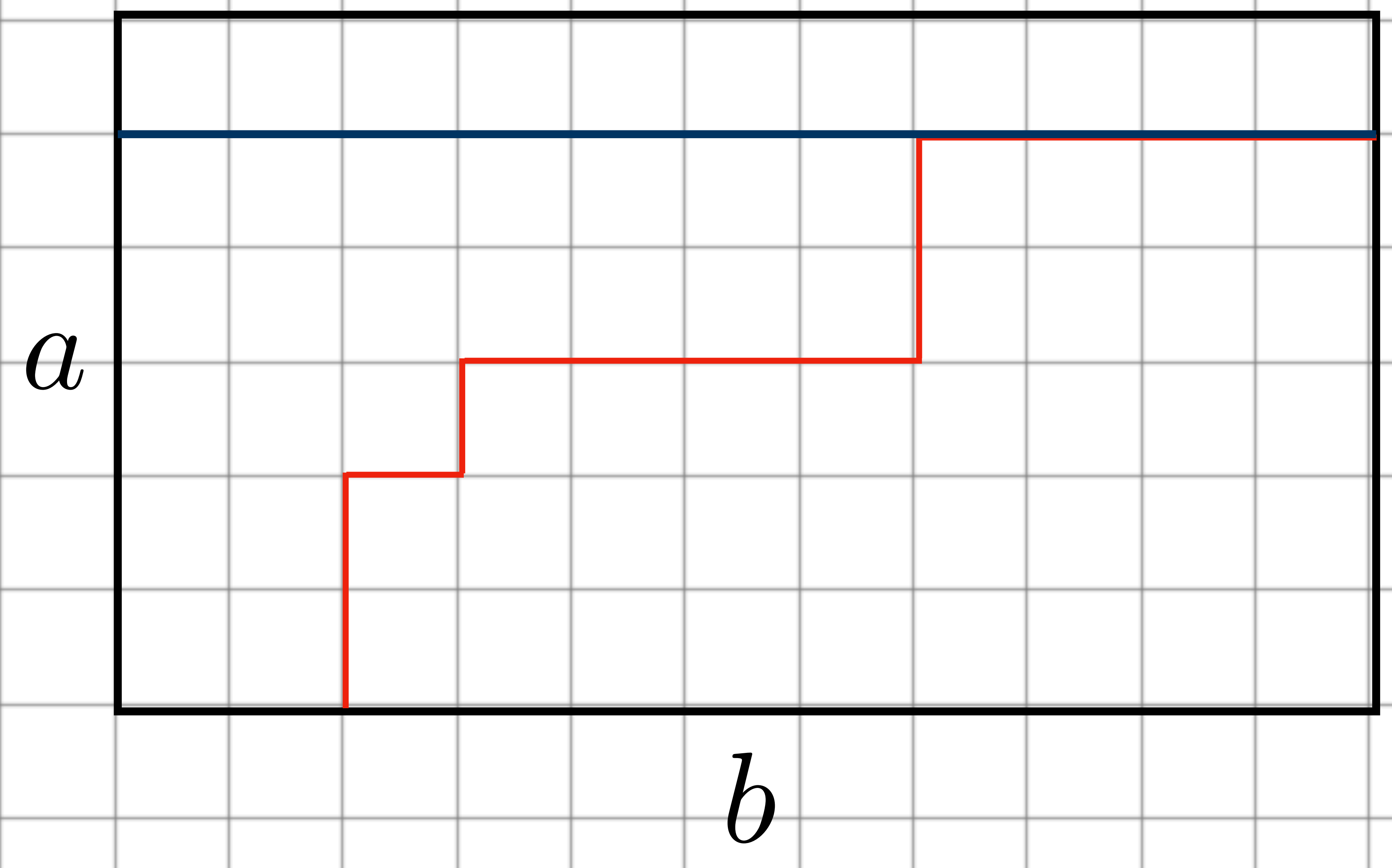
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof: Homework exercise if you like algebra

Inductive Step



$p(n; a, b-1)$



$p(n-b; a-1, b)$

Recursive Relation

$$p(n;a,b) = p(n;a,b-1) + p(n-b;a-1,b)$$

We get

$$\sum_{\lambda} q^{|\lambda|} = \sum_{n} p(n;a,b)\, q^n = \sum_{n} p(n;a,b-1)q^n + \sum_{n} p(n-b;a-1,b)q^n$$

$$= \sum_{n} p(n;a,b-1)q^n + \sum_{m} p(m;a-1,b)q^{m+b}$$

$$= \sum_{n} p(n;a,b-1)q^n + q^b \sum_{n} p(n;a-1,b)q^n$$

By inductive assumption

$$= \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix} + q^b \begin{bmatrix} a+b-1 \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}$$

Binomial Coefficients again

Exercise

$$\binom{a+b}{a} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{i+j}{i+j-1}$$

Box Formula

q-deformation works verbatim

$$\begin{bmatrix} a+b \\ a \end{bmatrix} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[i+j]}{[i+j-1]}$$

Now let us extend the box to infinity: $a, b \to \infty$

$$\sum_{\lambda} q^{|\lambda|} = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{[i+j]}{[i+j-1]} = \prod_{i=1}^{\infty} \frac{[i+1][i+2][i+3]\dots}{[i][i+1][i+2][i+3]\dots}$$

$$= \prod_{i=1}^{\infty} \frac{1}{[i]} = \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

Recall that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \dots$$

so

$$\frac{1}{1-z^k} = 1 + z^k + z^{2k} + z^{3k} + z^{4k} + \dots$$

Generating function

$$\mathbf{p}(z) = \frac{1}{\phi(z)} = \prod_{k=1}^{\infty} \frac{1}{1-z^k} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\dots}$$

$$\mathbf{p}(z) = (1 + z + z^2 + z^3 + \dots)(1 + z^2 + z^4 + z^6 + \dots)(1 + z^3 + z^6 + z^9 + \dots)\cdot\dots$$

$$\mathbf{p}(z) = \prod_{k=1}^{\infty}(1 + z^k + z^{2k} + z^{3k} + \dots)$$

$$\mathbf{p}(z) = (1+z+z^2+z^3+\dots)(1+z^2+z^4+z^6+\dots)(1+z^3+z^6+z^9+\dots)\cdot\dots$$

Now we need to collect terms in front of each power of z. Each term z^n in the resulting product will look like

$$z^{k_1}\cdot z^{2k_2}\cdot z^{3k_3}\cdot\dots\cdot z^{mk_m} = z^{k_1+2k_2+3k_3+\dots+mk_m}$$

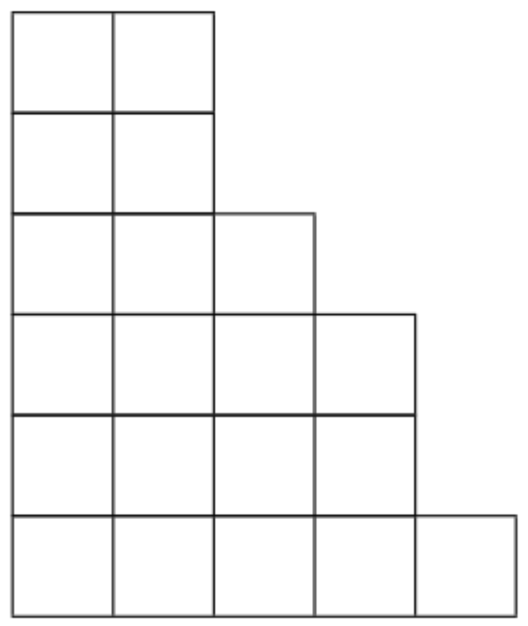
We want to count the number of such products with $k_1+2k_2+3k_3+\dots+mk_m = n$

$$n = k_1+2k_2+\dots+mk_m = \underbrace{1+\dots+1}_{k_1}+\underbrace{2+\dots+2}_{k_2}+\dots+\underbrace{m+\dots+m}_{k_m}$$

which is the number of partitions of n $\{\underbrace{m,\dots,m}_{k_m},\underbrace{m-1,\dots,m-1}_{k_{m-1}},\dots,\underbrace{2,\dots,2}_{k_2},\underbrace{1,\dots,1}_{k_1}\}$

consider partition $\{6,6,4,3,1\}$ of 20

$m=6$ and $k_6=2, k_5=0, k_4=1, k_3=1, k_2=0, k_1=1.$

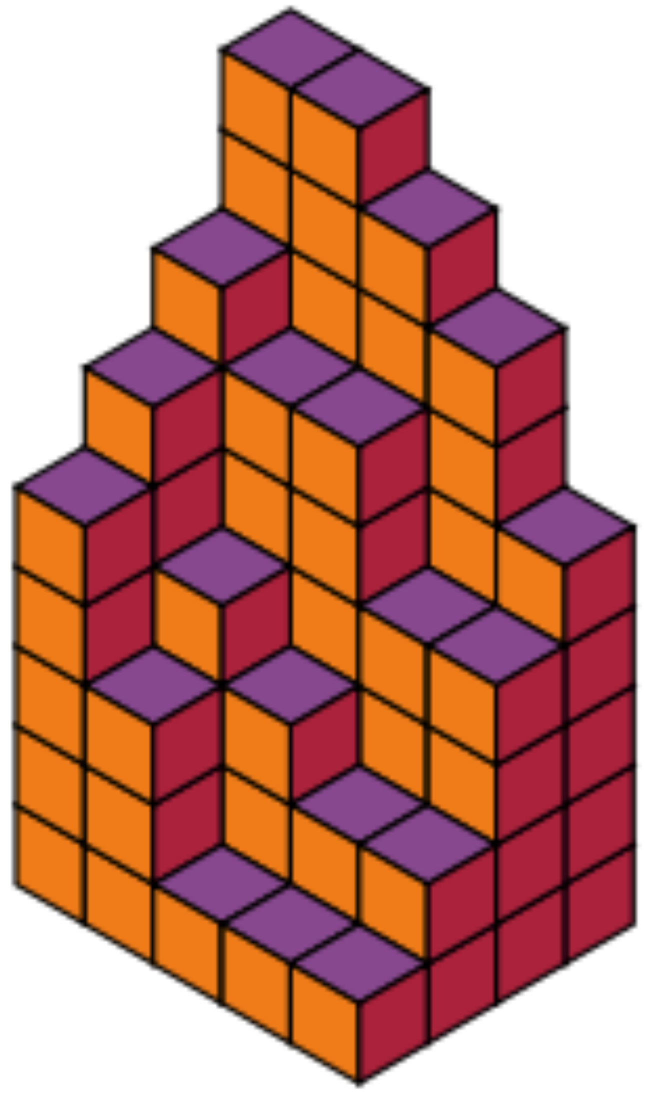


$$\mathbf{p}(z) = 1+p(1)z+p(2)z^2+p(3)z^3+p(4)z^4+$$

$$\mathbf{p}(z) = 1+z+2z^2+3z^3+5z^4+7z^5+11z^6+15z^7+22z^8+30z^9+42z^{10}+56z^{11}+77z^{12}+101z^{13}+\dots$$

Box Formula — 3d Upgrade!

$$\sum_{\pi \in \mathrm{Box}} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{[i+j+k-1]}{[i+j+k-2]}$$



$$\sum_{\forall \pi} q^{|\pi|} = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} \frac{[i+j++k-1]}{[i+j+k-2]}$$

$$= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{[i+j][i+j+1][i+j+2]\dots}{[i+j-1][1+j][i+j+1][i+j+2]\dots}$$

$$= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{[i+j-1]} = \prod_{i=1}^{\infty} \frac{1}{[i][i+1][i+2][i+3]\dots}$$

$$= \prod_{i=1}^{\infty} \frac{1}{[i[^{i}} = \prod_{i=1}^{\infty} \frac{1}{(1-q^{i})^{i}}.$$