The Wonderful World of Complex Numbers

1. Why complex numbers are needed?
2. Complex numbers in solving equations.
3. Paradoxes
4. Complex numbers in Rectangular Form ($a+bi$)
5. Operations of complex numbers
6. Parities in complex numbers
7. Complex prime numbers
8. Complex numbers Polar Form
9. Solve equations using complex numbers in Polar Form
10. Complex numbers in Exponential Form
11. Solve equations using complex numbers in Exponential Form

New sets of numbers are defined usually because of the necessity of solving certain kinds of equations.

Kronecker – 1886

"God created the Natural Numbers, everything else is the work of men"

Foundation Mathematicians like Godel – 20^{th} Century

"God created the number 1, everything else is the work of men"

Given the Natural Numbers: 1, 2, 3, 4, 5, …

0 (Zero) – The number 0 fulfills a central role in mathematics.

1. Additive Identity $0 + 8 = 8 + 0 = 8$
2. Position Holder CII +XCVIII = CC

   $102 + 98 = 200$
3. Solve Equations $x + 4 = 4$

Negative Integers – Negative integers fulfilled a central role in mathematics as the additive inverses of the integers, real numbers, and many other algebraic structures. $8 + (-8) = 0$

Negative integers also satisfied certain linear equations that involved addition and natural numbers such as $x + 4 = 3$.

Rational Numbers – Rational numbers fulfilled a central role in mathematics as the multiplicative inverses of the integers, real numbers, and many other algebraic structures (except 0). $8 \times \frac{1}{8} = 1$

Rational numbers also satisfied certain linear equations that involved multiplication and natural numbers such as $3 \times x = 2$. Rational numbers can be written as fractions of integers or as repeating decimals.

$$\frac{2}{11} = 0.181818\ldots$$

Algebraic Numbers – Any odd degree polynomial of integer coefficients has at least one algebraic zero. Any polynomial of integer coefficients has zeros that are either algebraic or complex numbers.

Examples:

(1) $\sqrt{2}$ is the solution for $x^2 - 2 = 0$.

(2) $1+\sqrt{3}$ is the solution for $x^2 - 2x - 2 = 0$.

(3) $\sqrt[3]{2}$ is the solution for $x^3 - 2 = 0$.

(4) π is not algebraic (transcendental).

Irrational Numbers – Any numbers that are not rational. There are some irrational numbers that are algebraic such as:

$$\sqrt{2},\ \sqrt{2}+\sqrt{5},\ \sqrt[3]{3}$$

And some that are not algebraic (or transcendental) such as:

$$\pi,\ e,\ e^{\pi},\ 2^{\sqrt{2}}$$

1. Natural Numbers 8
2. Zero 0
3. Negative Numbers -7
4. Rational Numbers $2/11$
5. Irrational Numbers $\sqrt{2}$
6. Real Numbers $0.010010001000001\ldots$
7. Algebraic Numbers $\frac{-1+\sqrt{5}}{2}$
8. Transcendental Numbers π

Irrational/Algebraic/Transcendental Numbers

$\sqrt{2}$ – Irrational/Algebraic

$2^{\sqrt{2}}$ – Irrational/Transcendental

$\sqrt{2}^{\sqrt{2}}$ **– Irrational/Transcendental**

$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ – Rational (= 2)

$\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ – Irrational

$\sqrt{2}^{\sqrt{2}^{\sqrt{2}\cdots}}$ – Rational (= 2)

$(\text{Irrational})^{\text{Irrational}}$ may be rational. E.g. $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$

In general, $(\text{algebraic not 0 or 1})^{\text{Irrational/algebraic}}$ = transcendental

(Hilbert Problem#7)

$(\sqrt{2})\hat{}\,(\log_{\sqrt{2}} 3)$ = rational (= 3) … $\log_{\sqrt{2}} 3$ cannot be algebraic.

This gets worse with complex numbers $i^i = 0.207879576\ldots$

We do not even know if the following numbers are irrational (let alone being algebraic):

$\pi + e$

πe

π / e

2^e

π^e

e^e

π^π

$\pi^{\sqrt{2}}$

$\ln(\pi)$

Real Numbers – All Rational plus Irrational Numbers.

SOLVING EQUATIONS

$$3x + 2 = 8$$

$$3x + 2 = 8$$

$$x = 2$$

Natural Number

$$5x + 2 = 2x + 2$$

$$5x + 2 = 2x + 2$$

Zero

$$5x + 20 = 2x + 2$$

$$5x + 20 = 2x + 2$$

$$x = -6$$

Negative Number

$$5x + 10 = 2x + 23$$

$$5x + 10 = 2x + 23$$

$$x = 13/3$$

Rational Number

$$x^2 - 10 = 2x$$

$$x^2 - 10 = 2x$$

$$x = 1 \pm \sqrt{11}$$

<u>Quadratic Formula</u>

Irrational Number

$$x^2 + 1 = 0$$

$$x^2 + 1 = 0$$

$$x = \pm i = \pm\sqrt{-1}$$

Imaginary Number

Imaginary Numbers

Some polynomials with integer coefficients have no real zeros such as $x^2 + 1 = 0$.

Solving $x^2 + 1 = 0$, we get some non–real numbers $x = \pm\sqrt{-1}$. We will label these new, non–real number as $i = \sqrt{-1}$ and the zeros for this equation are $\pm \boldsymbol{i}$. All numbers in the form of $\boldsymbol{bi}$ where $\boldsymbol{b}$ is a real number are called ***imaginary numbers***.

Properties of Imaginary Numbers

$i = \sqrt{-1}$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$

$i^5 = i$ $i^6 = -1$ $i^7 = -i$ $i^8 = 1$

$i^{239} = ????$ $-i$

$i^{122} = -1$

$\frac{1}{i} = \frac{1 \times i}{i \times i} = \frac{i}{-1} = -i$ (Checking: $-i^2 = 1$)

$\sqrt{i} = ???????$ **It turns out this is not an imaginary number.** $x^2 = i$

Paradoxes

Contradictions with imaginary and complex numbers:

1. $i = \sqrt{-1} = \sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}} = \frac{\sqrt{1}}{\sqrt{-1}} = \frac{1}{i}$.

So, $i^2 = 1$ but $i^2 = -1$.

Contradictions with imaginary and complex numbers:

2. $i^2 = (i^2)^{4/4} = (i^4)^{2/4} = [(i^2)^2]^{2/4}$

$$= [(-1)^2]^{2/4} = 1^{2/4} = 1.$$

Contradictions with imaginary and complex numbers:

3. $-1 = (-1)^1 = (-1)^{2/2} = [(-1)^2]^{1/2}$

$= (1)^{1/2} = 1.$

In complex number system, $(1)^{1/2}$ has two answers

$$(1)^{1/2} = \pm 1 \quad \text{because} \quad (\pm 1)^2 = 1$$

$$(-1)^{1/2} = \pm i \quad \textit{because} \quad (\pm i)^2 = -1$$

That is why, in real number system, we can use the notation $\sqrt{1}$ to indicate our choice of the "positive" root of 1. However, in complex numbers, there is no "***positive***".

It turns out we need more than just imaginery numbers.

New sets of numbers are defined usually because of the necessity of solving certain kind of equations.

$$x^2 + 2x + 5 = 0$$

$$x^2 + 2x + 5 = 0$$

$$x = -1 \pm 2i$$

Complex Number

$$a + bi$$

a = ***Real Part***, b = ***Imaginery Part***

$$x^3 - 1 = 0$$

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$$x = 1, -1/2 \pm \sqrt{3}/2\ i$$

$$x^4 - 1 = 0$$

$$x^4 - 1 = 0$$

$$(x+1)(x-1)(x^2+1) = 0$$

$$x = \pm 1, \pm i$$

$$x^5 - 1 = 0$$

$$x^5 - 1 = 0$$

$$(x-1)(x^4+x^3+x^2+x+1) = 0$$

$$x = 1, ???$$

Fundamental Theorem of Algebra

(proved by Carl Gauss in 1799) states that

Every polynomial equation of degree *n* with complex number coefficients has *n* roots or solutions or zeros in complex numbers (some may be redundant or duplicate solutions).

$$x^{90} - 1 = 0$$

$$x^{90} - 1 = 0$$

$$(x^{45}-1)(x^{45}+1) = 0$$

$$x^2 = 9$$

$$x^2 = 9$$

$$x = \pm 3$$

$$2^x = 8$$

$$2^x = 8$$

$$x = 3$$

$$2^x = 1$$

$$2^x = 1$$

$$x = 0$$

$$1^x = 2$$

$$1^x = 2$$

$$x = ????????$$

What about

$$\sqrt{i}$$

$$2^i$$

$$1^i$$

$$i^i$$

$$(-i)^i$$

$$\sqrt[i]{i}$$

$$\sqrt[4+5i]{2-3i}$$

Complex Numbers = Numbers in the form of $\boldsymbol{z = a + bi}$ where a and b are real numbers. We call

$\boldsymbol{a = R(z)}$ the **<u>real part</u>** of z and

$\boldsymbol{b = I(z)}$ the **<u>imaginary part</u>** of z.

$$(\boldsymbol{z = R(z)+I(z)i})$$

In this case, polynomials with real coefficients and degree n have n complex zeros (some zeros may be real and some may be duplicate zeros.)

Complex Numbers:

$$\boldsymbol{z = a + bi} \quad (i^2 = -1)$$

$$(3-5i) + (-2+8i) = 1+3i$$

$$(3-5i) \times (-2+8i) = (-6-40i^2) + (24i+10i)$$

$$= 34+34i$$

$$(3+5i) \times (3-5i) = (9+25) + (15i-15i)$$

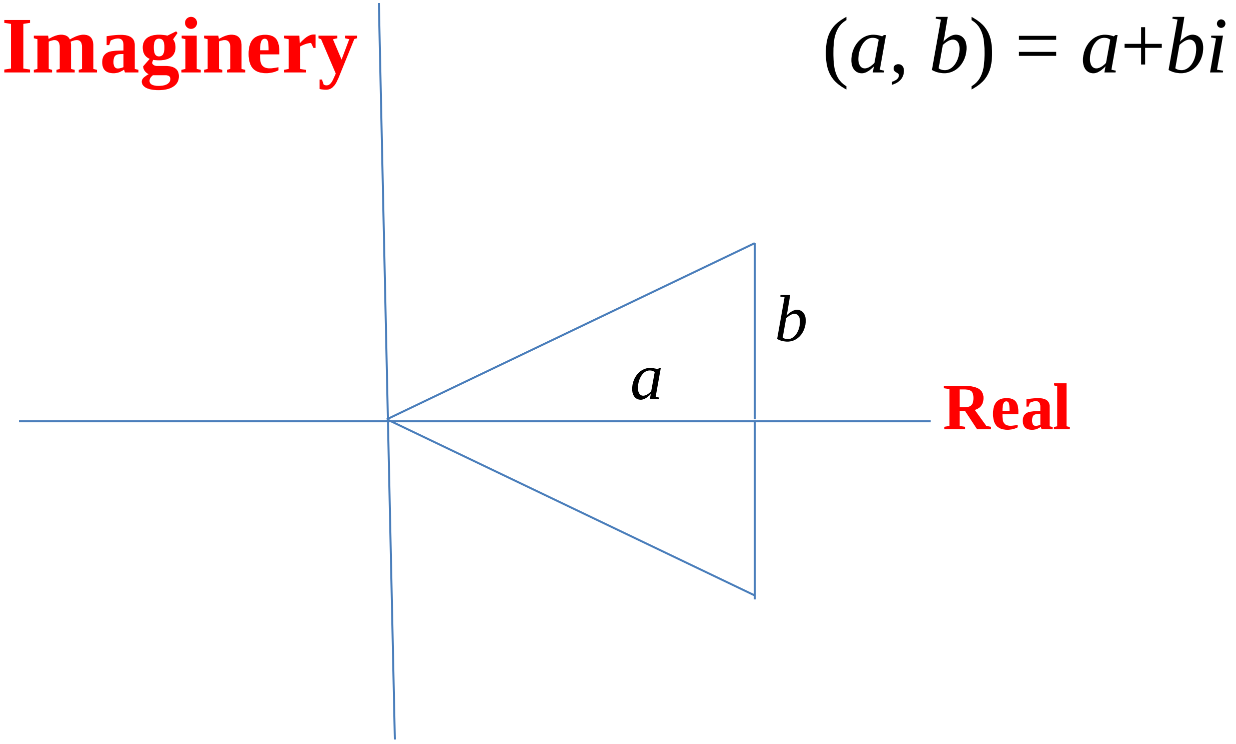
$$= 34 \quad (\textbf{\textit{Conjugates}})$$

$$\frac{-2+8i}{3+5i} = \frac{(-2+8i)(3-5i)}{(3+5i)(3-5i)} = \frac{34+34i}{34} = 1+i$$

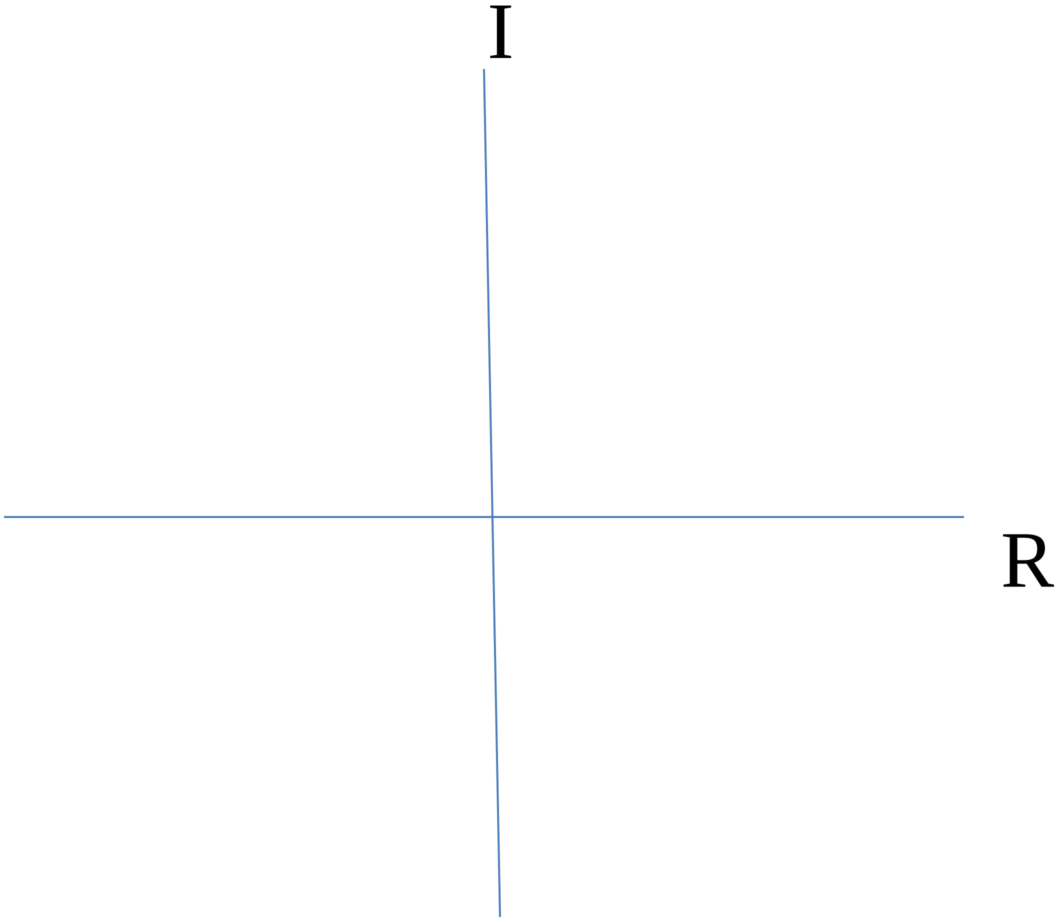
Complex Numbers

We label $z^* = a - bi$ as the "***conjugate***" of z. z^* is important because $z^* \times z = z \times z^* = a^2 + b^2$ which is a real number.

We can represent complex numbers graphically on a *x*–*y* coordinate system where point (a, b) represents the complex number $a+bi$. We call this the ***rectangular form*** of complex numbers.

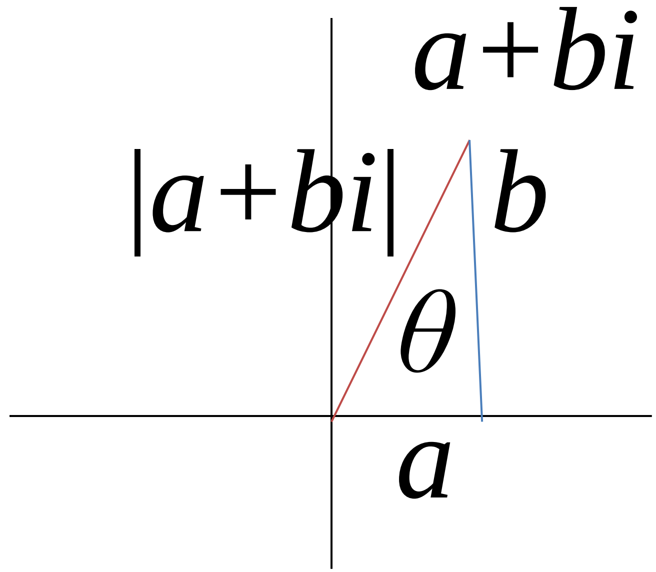


Examples:



$4,\ i,\ -i,\ 3+4i,\ -2+i,\ -3-2i,\ 4-3i$

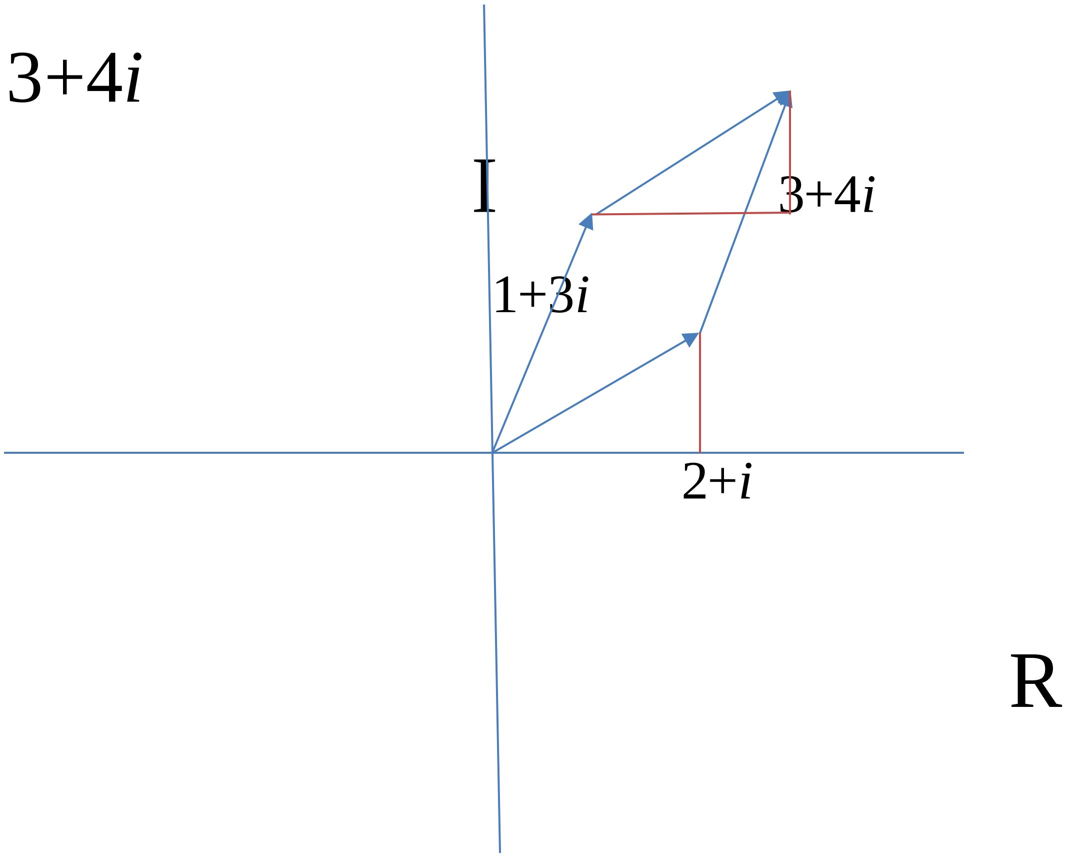
Every complex number $a+bi$ in rectangular form is determined by:



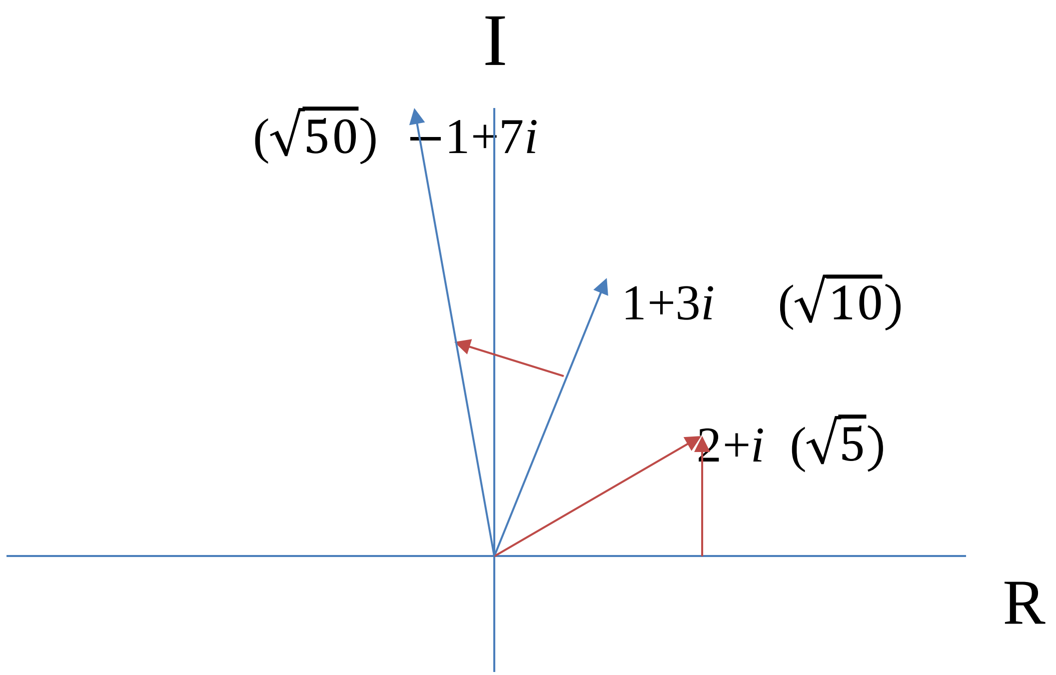
1. Length: $|a+bi| = a^2+b^2$.
2. Angle: $\theta(a+bi)$

Adding 2 complex numbers graphically.

$(2+i) + (1+3i) = 3+4i$



Multiply 2 complex numbers graphically.



$\theta(-1+7i) = \theta((2+i)\times (1+3i)) = \theta(2+i) + \theta(1+3i)$

$|-1+7i| = |(2+i)\times (1+3i)| = |(2+i)| \times |(1+3i)|$

$\sqrt{i}$ = ??????? It turns out the answer is

not an imaginary number.

Try $x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$ $\qquad$ $\boldsymbol{x^2 = i}$

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right)^2 = \frac{1}{2} + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)i + \left(-\frac{1}{2}\right) = i$$

i^i = ??????? The answer is a real number.

$\sqrt[i]{i}$ = ??????? The answer is a real number.

Are there any “**even**” or “**odd**” complex numbers?

Are there any “**even**” or “**odd**” complex numbers?

1. No even or odd real numbers.
2. No even or odd complex numbers.
3. Integers in real numbers
4. **Gaussian Integers** in complex numbers

$2+2i$ even ?

$3+3i$ odd ?

$2+2i$ even ?

$3+3i$ odd ?

What about $2+3i$?

$3+2i$?

2 or 3 ?

$2i$ or $3i$?

Before we answer these questions, we must identify “even–ness” and “odd–ness”?

Even = divisible by 2 does **not** quite work since complex numbers consist of 2 parts.

If this is the case, then only **(even)+*i*(even)** are even numbers…all the others are odd numbers.

Let’s look at the “Even-ness” and “Odd-ness” in integers.

Even + Even	= **Even**
Odd + Odd	= **Even**
Even + Odd	= **Odd**
Odd + Even	= **Odd**
Even × Even	= **Even**
Even × Odd	= **Even**
Odd × Even	= **Even**
Odd × Odd	= **Odd**

We would like to keep these properties for Gaussian Integers.

$2 + 2i$ is **Even** and $3+3i$ is **Odd**.

$(2+2i)+(4+4i) = 6+6i$ (Even + Even = **Even**)

$(2+2i)\times(4+4i) = 0+16i$ (Even × Even = **Even**)

$(2+2i)+(3+3i) = 5+5i$ (Even + Odd = **Odd**)

$(2+2i)\times(3+3i) = 0+12i$ (Even × Odd = **Even**)

$(3+3i)+(1+1i) = 4+4i$ (Odd + Odd = **Even**)

(Even)+(Even)i = **Even** (Odd)+(Odd)i = **Odd**

2 + 2i is **<u>Even</u>** 3 + 3i is **<u>Odd</u>**.

4 + 6i is **<u>Even</u>** 5 + 7i is **<u>Odd</u>**.

(2+2i)+(4+4i) = 6+6i (Even + Even = **Even**)

(2+2i)×(4+4i) = 0+16i (Even × Even = **Even**)

(2+2i)+(3+3i) = 5+5i (Even + Odd = **Odd**)

(2+2i)×(3+3i) = 0+12i (Even × Odd = **Even**)

(3+3i)+(1+1i) = 4+4i (Odd + Odd = **Even**)

(3+3i)×(1+1i) = 0+6i (Odd × Odd = **Even** ???)

First, we want to make sure Gaussian Integers are consistent with Real Integers which means that the numbers 1, 3, 5, 7, …1+0*i*, 3+0*i* should be **odd** and 2, 4, 6, 8, … 2+0*i*, 4+0*i* should be **even**.

Hint: It must also be consistent with realintegers.

For example, $4 = 4 + 0i$ must be **even** and

$3 = 3 + 0i$ must be **odd**.

From the fact that $1+0i$ (one odd and one even) is **odd** and $2+0i$ (both even) is **even**, you may guess that

$(Odd)+(Even)i$ is **odd**

and

$(Even)+(Even)i$ is **even**

(Odd)+(Even)i is **odd**

(Even)+(Even)i is **even**

What about

(Even)+(Odd)i and

(Odd)+(Odd)i ?

In that case, numbers like (Odd)+(Odd)i cannot be **odd** since $(1+3i)\times(1-3i) = 1 + 9 = 10$ is **even**.

[(Odd)+(Odd)*i*]×[(Odd)+(Odd)*i*] = [(Odd-Odd)+(Odd+Odd)*i*]

= (Even)+(Even)*i*

[(Even)+(Even)*i*]×[(Even)+(Even)*i*] = [(Even-Even)+(Even+Even)*i*]

= (Even)+(Even)*i*,

these types of Gaussian Integers should be even.

In that case, numbers like (Even)+(Odd)i cannot be **even** since $(4+3i)\times(4-3i) = 16-9 = 7$ is **odd**.

[(Odd)+(Even)*i*]×[(Even)+(Odd)*i*] = [(Even-Even)-(Even+Odd)*i*]

= (Even)+(Odd)*i* (Odd × Odd = Odd)

[(Odd)+(Even)*i*]+[(Even)+(Odd)*i*] = [(Odd+Even)+(Even+Odd)*i*]

= (Odd)+(Odd)*i* (Odd + Odd = Even)

these types of complex numbers should be odd.

If a and b are of <u>same parity</u>, then $a+bi$ is <u>even</u>.

$3+5i$, $4+2i$, $10-8i$, $3+3i$, $4+4i$

$8+0i = 8$

(or $a+b$ = even+even or odd+odd = <u>even</u>)

If a and b are of <u>different parity</u>, then $a+bi$ is <u>odd</u>.

$3+4i$, $4+5i$, $10-7i$, $7+0i = 7$

(or $a+b$ = even+odd or odd+even = <u>odd</u>)

Numbers with complex factors.

Because $(3+2i)(8-5i) = 34+i$, both $(3+2i)$ and $(8-5i)$ are "factors" of $(34+i)$.

Gaussian Integers do not have unique factoring.
Therefore, we cannot talk about complex prime numbers.

We still can apply complex factors to regular prime numbers.

Before, we say 2 is a prime number since factors of 2 can only be 1 and 2. However, if we consider Gaussian complex factors, $2 = (1+i)(1-i)$ so 2 is a real prime but not a complex prime.

$2 = (1+i)(1-i)$ so 2 is a real prime but not a complex prime.

What about 3? 3 is a real prime, but is 3 a complex prime?

What about 5 or 7 or 11 or 13 or ….?

We are all familiarized with Prime Numbers:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, …

Let us consider positive integers with Gaussian complex factors. Then, some of the above prime numbers are not really prime numbers (they are composite numbers)

For examples:

$2 = (1+i)(1-i)$

$5 = (1+2i)(1-2i)$

$13 = (2+3i)(2-3i)$

$17 = (4+i)(4-i)$

.

.

.

The bold faced “prime numbers” below are the true “Gaussian complex” prime numbers.

2, **3**, 5, **7**, **11**, 13, 17, **19**, **23**, 29, **31**, 37, 41, **43**, **47**, 53, **59**, 61, **67**, **71**, 73…

Only about half of the original prime numbers are Gaussian primes.

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$2 = (1+i)(1-i)$

$5 = (1+2i)(1-2i)$

$13 = (2+3i)(2-3i)$

$17 = (4+i)(4-i)$

$29 = (5+2i)(5-2i)$

$37 = (6+i)(6-i)$

$41 = (5+4i)(5-4i)$

$53 = (7+2i)(7-2i)$

$61 = (6+5i)(6-5i)$

$73 = (8+3i)(8-3i)$

$89 = (8+5i)(8-5i)$

$97 = (9+4i)(9-4i)$

2, **3**, 5, **7**, **11**, 13, 17, **19**, **23**, 29, **31**, 37, 41, **43**, **47**, 53, **59**, 61, **67**, **71**, 73…

$2 = (1+i)(1-i)$

$5 = (1+2i)(1-2i)$

$13 = (2+3i)(2-3i)$

$17 = (4+i)(4-i)$

$29 = (5+2i)(5-2i)$

$37 = (6+i)(6-i)$

$41 = (5+4i)(5-4i)$

$53 = (7+2i)(7-2i)$

$61 = (6+5i)(6-5i)$

$73 = (8+3i)(8-3i)$

$89 = (8+5i)(8-5i)$

$97 = (9+4i)(9-4i)$

2, **3**, 5, **7**, **11**, 13, 17, **19**, **23**, 29, **31**, 37, 41, **43**, **47**, 53, **59**, 61, **67**, **71**, 73…

What are the characteristics for Gaussian Primes?

If ***p*** is a prime but not a Gaussian prime,
then ***p*** = product of 2 complex **conjugates**.

If $\boldsymbol{p}$ is a prime (except 2) but not a Gaussian prime, then $\boldsymbol{p} \equiv 1 \pmod{4}$.

(When p is divided by 4, it has a remainder of 1)

If ***p*** is a <u>prime</u> (except 2) but **not** a <u>Gaussian prime</u>, then ***p*** ≡ 1 (mod 4).

Consider 4*q*, 4*q*+1, 4*q*+2, 4*q*+3. Then any integer squared must one of the following:

$(4q)^2 \equiv 0 \pmod{4}$

$(4q+1)^2 \equiv 1 \pmod{4}$

$(4q+2)^2 \equiv 0 \pmod{4}$

$(4q+3)^2 \equiv 1 \pmod{4}$

So, any integer squared must have either 0 or 1 as remainder when divided by 4. That means, in order for a prime number ***p*** <u>**not**</u> be a <u>Gaussian prime</u>, $p = (a+bi)(a-bi) = a^2+b^2$ and ***p*** must have remainder of **1** when divided by 4 or a <u>Gaussian prime</u> must have a remainder of **3** when divided by 4.

2, 5, 13, 17, 29, 37, 41, …

Any prime number $p \equiv 1 \pmod 4$ **Non-Gaussian Prime**

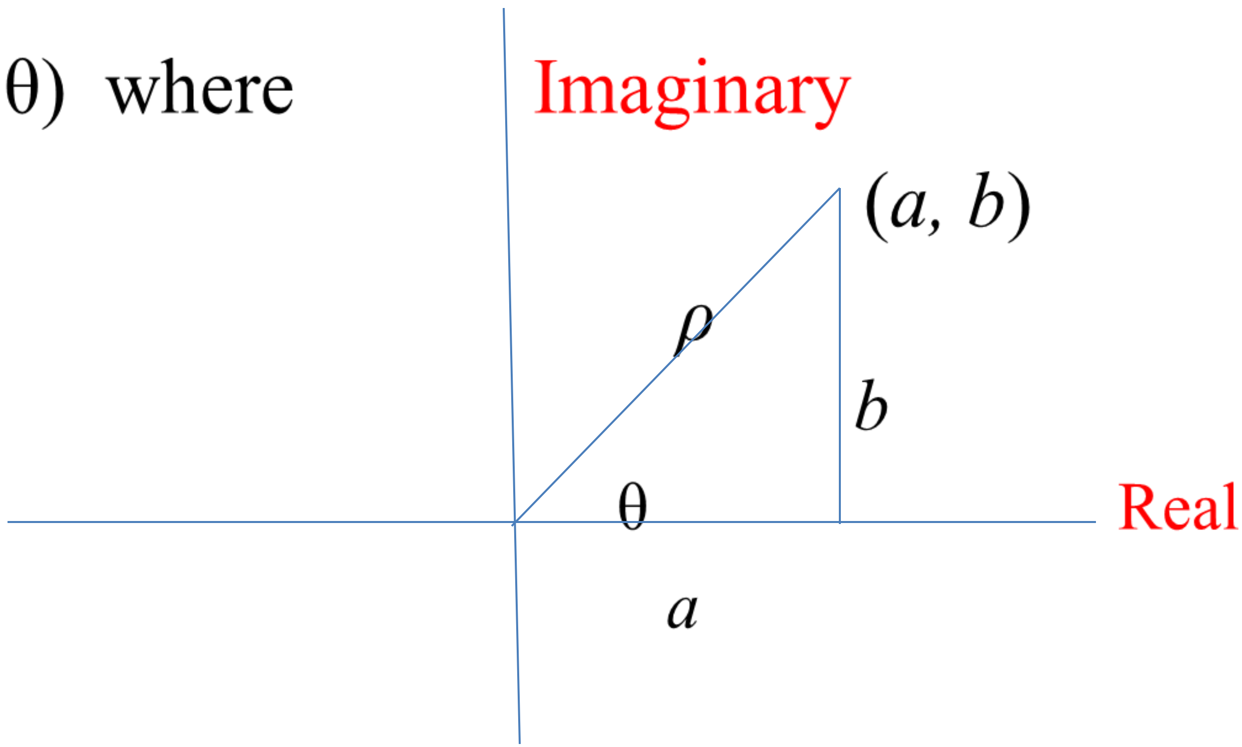
3, 7, 11, 19, 23, 31, 43, 47, …

Any prime number $p \equiv 3 \pmod 4$ **Gaussian Prime**

We can also use “radius” and “angle” to represent complex numbers graphically (in **Polar Form**).

Radius = ρ = $\sqrt{a^2+b^2}$. In this case, every complex number can be written using its “radius” and “angle”.

$a+bi = (\rho, \theta)$ where



One time around the circle is $360°$

Repeats every $360°$

So, $1440° \sim 1080° \sim 720° \sim -360° \sim 360°$

$540° \sim 180° \sim -180°$

$450° \sim 90° \sim -270°$

$765° \sim 405° \sim 45° \sim -315°$

$750° \sim 390° \sim 30° \sim -330°$

$780° \sim 420° \sim 60° \sim -300°$

$630° \sim 270° \sim -90° \sim -450°$

What exactly is (ρ, θ) ?

$$\mathbf{1} = (1, 0°) = (1, 360\ °) = (1, -360°)$$

$$\boldsymbol{i} = (1, 90°) = (1, 450°) = (1, -270°)$$

$$\mathbf{-1} = (1, 180°) = (1, 540°) = (1, -180°)$$

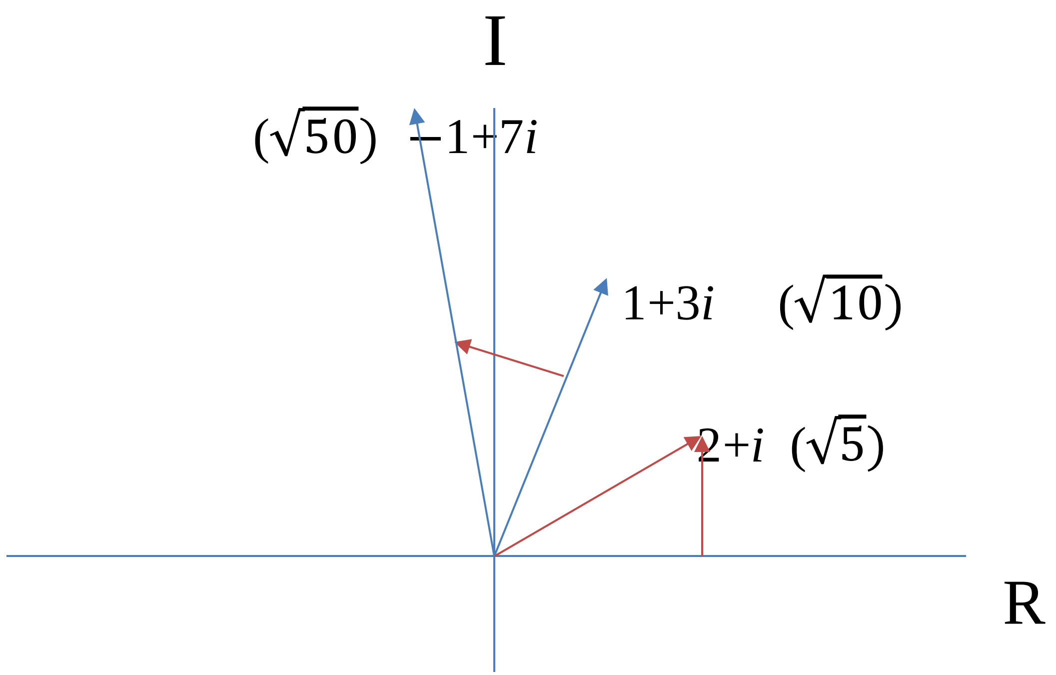
$$-\boldsymbol{i} = (1, 270°) = (1, -90°) = (1, 630°)$$

$$\mathbf{1} + \boldsymbol{i} = (\sqrt{2}, 45°) = (\sqrt{2}, 405°) = (\sqrt{2}, -315°)$$

$$\mathbf{1} + \sqrt{\mathbf{3}}\boldsymbol{i} = (2, 60°) = (2, 420°) = (2, -300°)$$

$$-\sqrt{\mathbf{3}} - \boldsymbol{i} = (2, 210°) = (2, 570°) = (2, -150°)$$

Multiply 2 complex numbers graphically.



$\theta(-1+7i) = \theta((2+i)\times (1+3i)) = \theta(2+i) + \theta(1+3i)$

$|-1+7i| = |(2+i)\times (1+3i)| = |(2+i)| \times |(1+3i)|$

$$(a + bi)^2 = (\rho, \theta)^2 = (\rho^2, 2\theta)$$

$$(a + bi)^3 = (\rho, \theta)^3 = (\rho^3, 3\theta)$$

$$\vdots$$

$$(a + bi)^n = (\rho, \theta)^n = (\rho^n, n\theta)$$

$$(a + bi)^{1/2} = (\rho, \theta)^{1/2} = (\rho^{1/2}, \theta/2)$$

$$(a + bi)^{1/n} = (\rho, \theta)^{1/n} = (\rho^{1/n}, \theta/n)$$

For example, there are 2 answers to square roots of 1.

$(x^2 = 1)$

1 is located at angle $\theta = 0°$ and radius $\rho = 1$.

So, $1 = (1, 0° + 360° n)$ $n = 0, 1, 2, 3, \ldots$

Therefore, $1^{1/2} = (1, 0° + 180° n)$ $n = 0, 1, 2, 3, \ldots$

$n = 0$ $(1, 0°) = \mathbf{1}$

$n = 1$ $(1, 0° + 180°) = \mathbf{-1}$

$n = 2$ $(1, 0° + 360°) = (1, 0°)$ repeating cycle of 2.

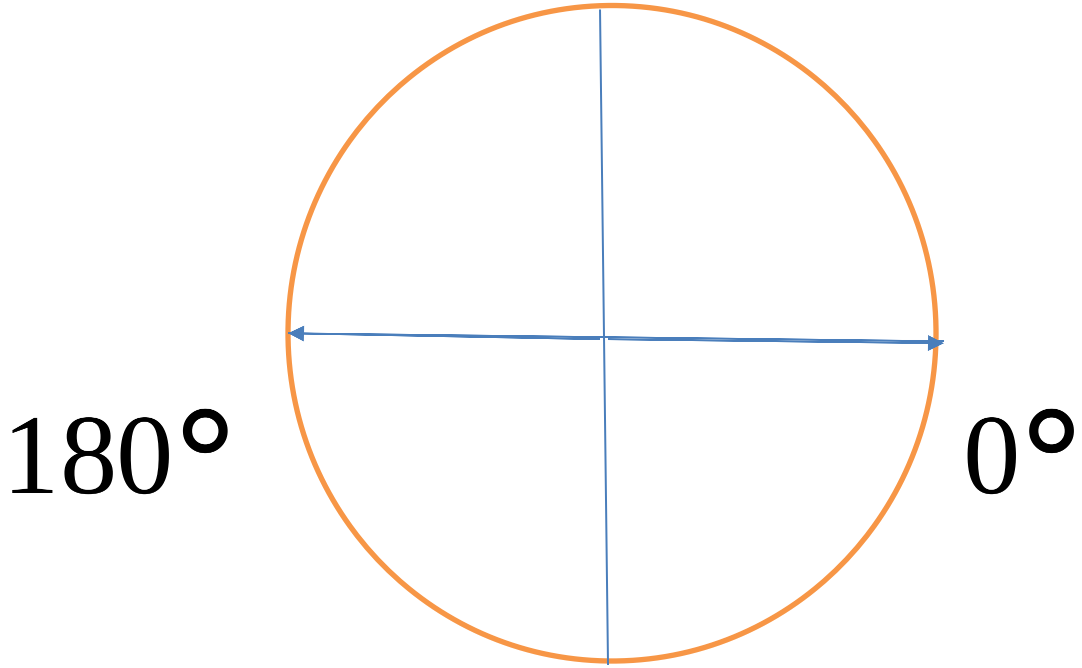
So, we have two answers for square roots of 1, $(1)^{1/2} = \pm 1$.

(In algebra of real numbers, we define $\sqrt[2]{1} = +1$.)

These represent the 2 points on the unit circle at angles 0°, 180°, each separated by 360/2 = 180°.

$n = 0$ $(1, 0°)$ $(x^2 = 1)$

$n = 1$ $(1, 180°)$



For example, there are 4 answers to 4th roots of 1.
$(x^4 = 1)$

1 is located at angle $\theta = 0°$ and radius $\rho = 1$.

So, $\mathbf{1 = (1,\ 0°+360°}\boldsymbol{n}\mathbf{)}$ $n = 0, 1, 2, 3, \ldots$

Therefore, $\sqrt[4]{1} = 1^{1/4} = (1,\ 0°+90°n)$ $n = 0, 1, 2, 3, \ldots$

$n = 0$ $(1,\ 0°) = \mathbf{1}$

$n = 1$ $(1,\ 0°+90°) = \boldsymbol{i}$

$n = 2$ $(1,\ 0°+180°) = \mathbf{-1}$

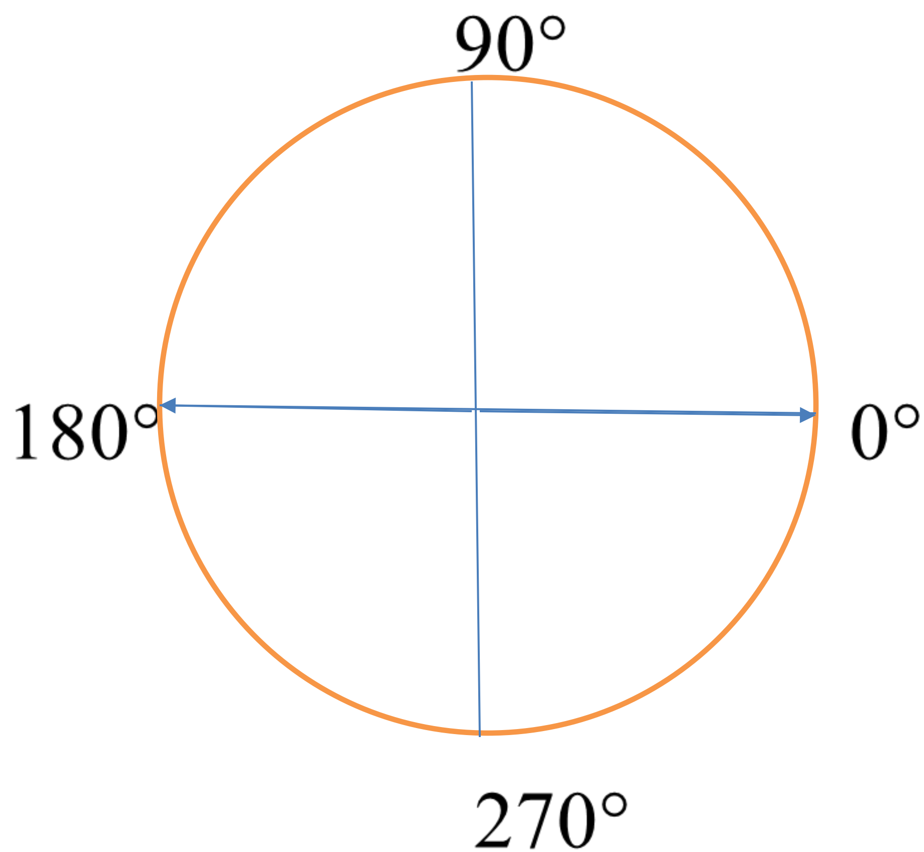
$n = 3$ $(1,\ 0°+270°) = \boldsymbol{-i}$

$n = 4$ $(1,\ 0°+360°) = (1,\ 0°) = \mathbf{1}$ repeating cycle of 4.

So, we have four answers for 4th roots of 1.

These represent the 4 points on the unit circle at angles 0°, 90°, 180°, 270°, each separated by 360/4 = 90°.

$n = 0$	$(1, 0°) = \mathbf{1}$	$(x^4 = 1)$
$n = 1$	$(1, 0°+90°) = \boldsymbol{i}$	
$n = 2$	$(1, 0°+180°) = \mathbf{-1}$	
$n = 3$	$(1, 0°+270°) = \boldsymbol{-i}$	



For example, there are 3 answers to cube roots of i.

$(x^3 = i)$

i is located at angle $\theta = 90°$ and radius $\rho = 1$.

So, $\boldsymbol{i = (1, 90°+360°n)}$ $n = 0, 1, 2, 3, \ldots$

Therefore, $\sqrt[3]{i} = i^{1/3} = (1, 30°+120°n)$ $n = 0, 1, 2, 3, \ldots$

$n = 0$ $(1, 30°)$

$n = 1$ $(1, 30°+120°) = (1, 150°)$

$n = 2$ $(1, 30°+240°) = (1, 270°)$

$n = 3$ $(1, 30°+360°) = (1, 30°)$ repeating cycle of 3.

So, we have three answers for $\sqrt[3]{i}$.

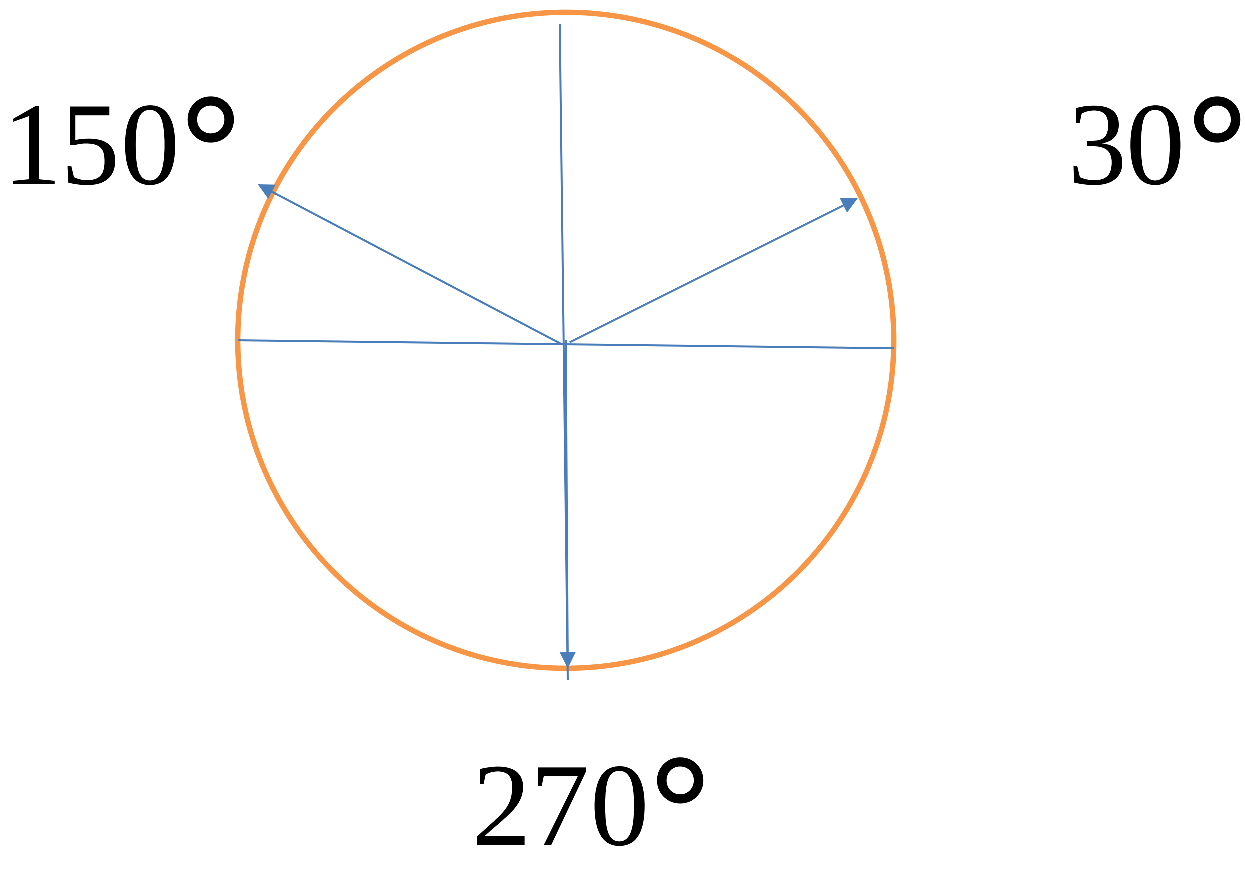
These represent the 3 points on the unit circle at angles 30°, 150°, 270° each separated by 360/3 = 120°.

$$\sqrt[3]{i} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -i$$

$n = 0$ $(1, 30°)$ $(x^3 = i)$

$n = 1$ $(1, 150°)$

$n = 2$ $(1, 270°)$



For example, there are 2 answers to square roots of $-i$.

$-i$ is located at angle $\theta = 270°$ and radius $\rho = 1$.

So, $-i = (1, 270° + 360° n)$ $n = 0, 1, 2, 3, \ldots$

Therefore, $\sqrt{-i} = (1, 135° + 180° n)$ $n = 0, 1, 2, 3, \ldots$

$n = 0$ $(1, 135°)$

$n = 1$ $(1, 135° + 180°) = (1, 315°)$

$n = 2$ $(1, 135° + 360°) = (1, 495°) = (1, 135°)$

repeating cycle of 2.

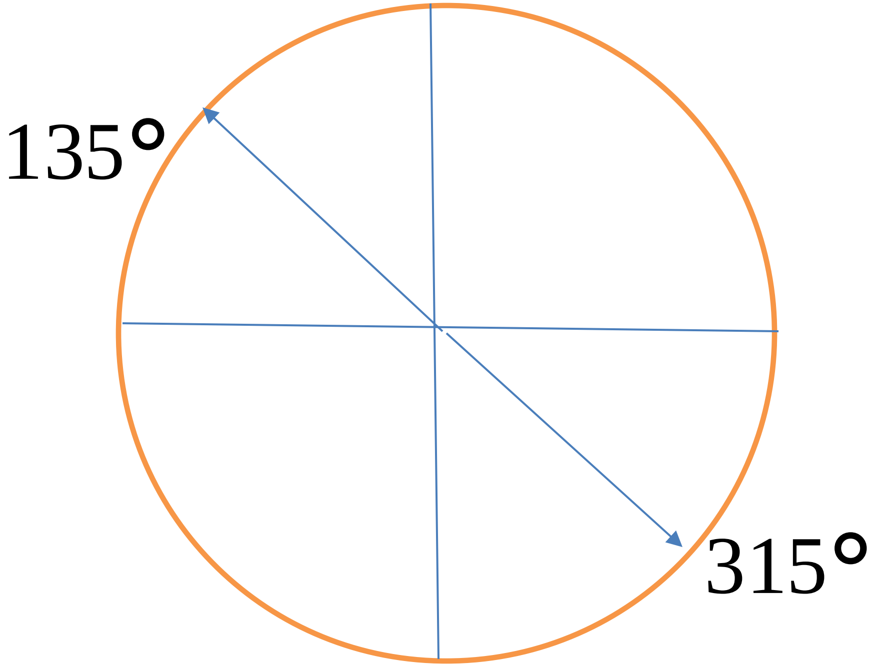
So, we have two answers for square roots of $(-i)$.

These represent the 2 points on the unit circle at angles 135°, and 315° separated by 360/2 = 180°.

$$\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) \quad \text{or} \quad \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$$

$n = 0 \quad (1, 135°)$

$n = 1 \quad (1, 315°)$



$\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right)$ or $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right)$

$$(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}+\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}i)^2=(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}+\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}i)(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}+\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}i)=(\frac{1}{2}-\frac{1}{2})+(-\frac{1}{2}-\frac{1}{2})i=-i$$

$$(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}i)^2=(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}i)(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}i)=(\frac{1}{2}-\frac{1}{2})+(-\frac{1}{2}-\frac{1}{2})i=-i$$

For example, there are 2 answers to square roots of $-1+\sqrt{3}i$.

$(x^2 = -1 + \sqrt{3}i)$

$-1+\sqrt{3}i$ is located at angle $\theta = 120°$ and radius $\rho = 2$.

So, **$-1 + \sqrt{3}i = (2, 120°+360°n)$** $n = 0, 1, 2, 3,\ldots$

Therefore, $\sqrt{-1+\sqrt{3}i} = (-1 + \sqrt{3}i\,)^{½} = (\sqrt{2}, 60°+180°n)$ $n = 0, 1, 2,$ $3,\ldots$

$n = 0$ $(\sqrt{2}, 60°)$

$n = 1$ $(\sqrt{2}, 240°)$

$n = 2$ $(\sqrt{2}, 420°) = (\sqrt{2}, 60°)$

repeating cycle of 2.

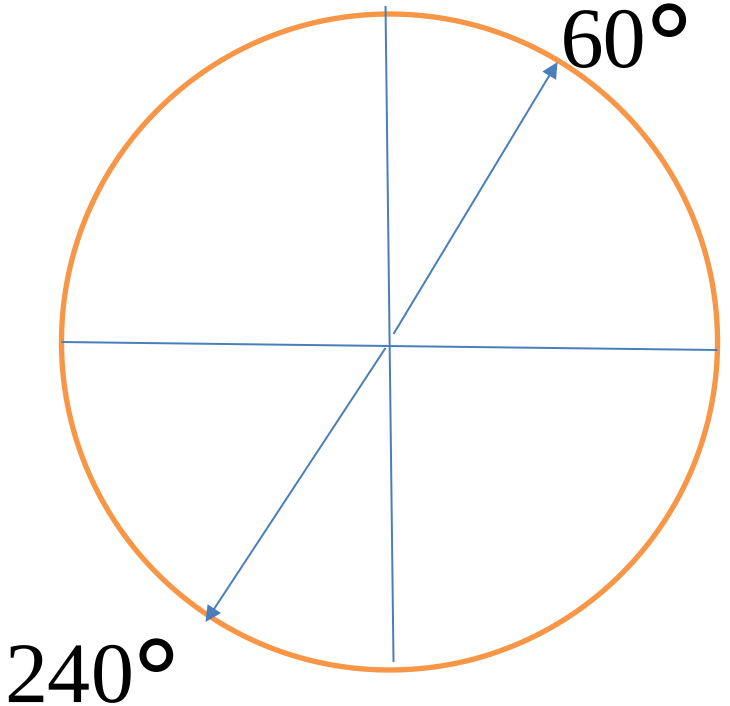
So, we have two answers for $\sqrt{-i}$.

These represent the 2 points on the unit circle at angles 60°, and 240° separated by 360/2 = 180°.

$$\sqrt{-1+\sqrt{3}i} = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \quad \text{or} \quad \sqrt{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$n = 0 \qquad (\sqrt{2}, 60°)$

$n = 1 \qquad (\sqrt{2}, 240°)$



For example, there are 5 answers to fifth roots of $-1+\sqrt{3}i$.

$(x^5 = -1 + \sqrt{3}i)$

$-1+\sqrt{3}i$ is located at angle $\theta = 120°$ and radius $\rho = 2$.

So, $-1+\sqrt{3}i = (2, 120°+360°n)$ $n = 0, 1, 2, 3, \ldots$

Therefore, $\sqrt[5]{-1+\sqrt{3}i} = (-1+\sqrt{3}i\,)^{1/5} = (\sqrt[5]{2}, 24°+72°n)$ $n = 0, 1, 2, 3, \ldots$

$n = 0$ $(\sqrt[5]{2}, 24°)$

$n = 1$ $(\sqrt[5]{2}, 96°)$

$n = 2$ $(\sqrt[5]{2}, 168°)$

$n = 3$ $(\sqrt[5]{2}, 240°)$

$n = 4$ $(\sqrt[5]{2}, 312°)$

$n = 5$ $(\sqrt[5]{2}, 384°) = (\sqrt[5]{2}, 24°)$

repeating cycle of 5.

These represent the 5 points on the unit circle at angles 24°, 96°, 168°, 240°, and 312° separated by 360/5 = 72°.

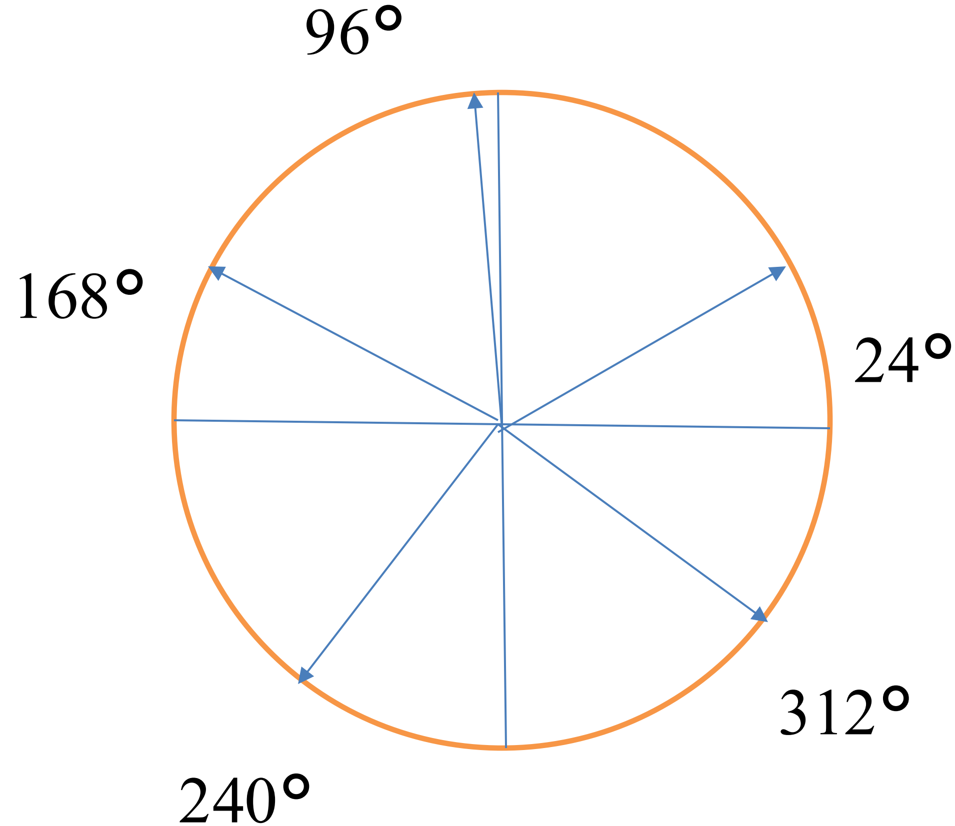
$n = 0$ $(, 24°)$

$n = 1$ $(, 96°)$

$n = 2$ $(, 168°)$

$n = 3$ $(, 240°)$

$n = 4$ $(, 312°)$

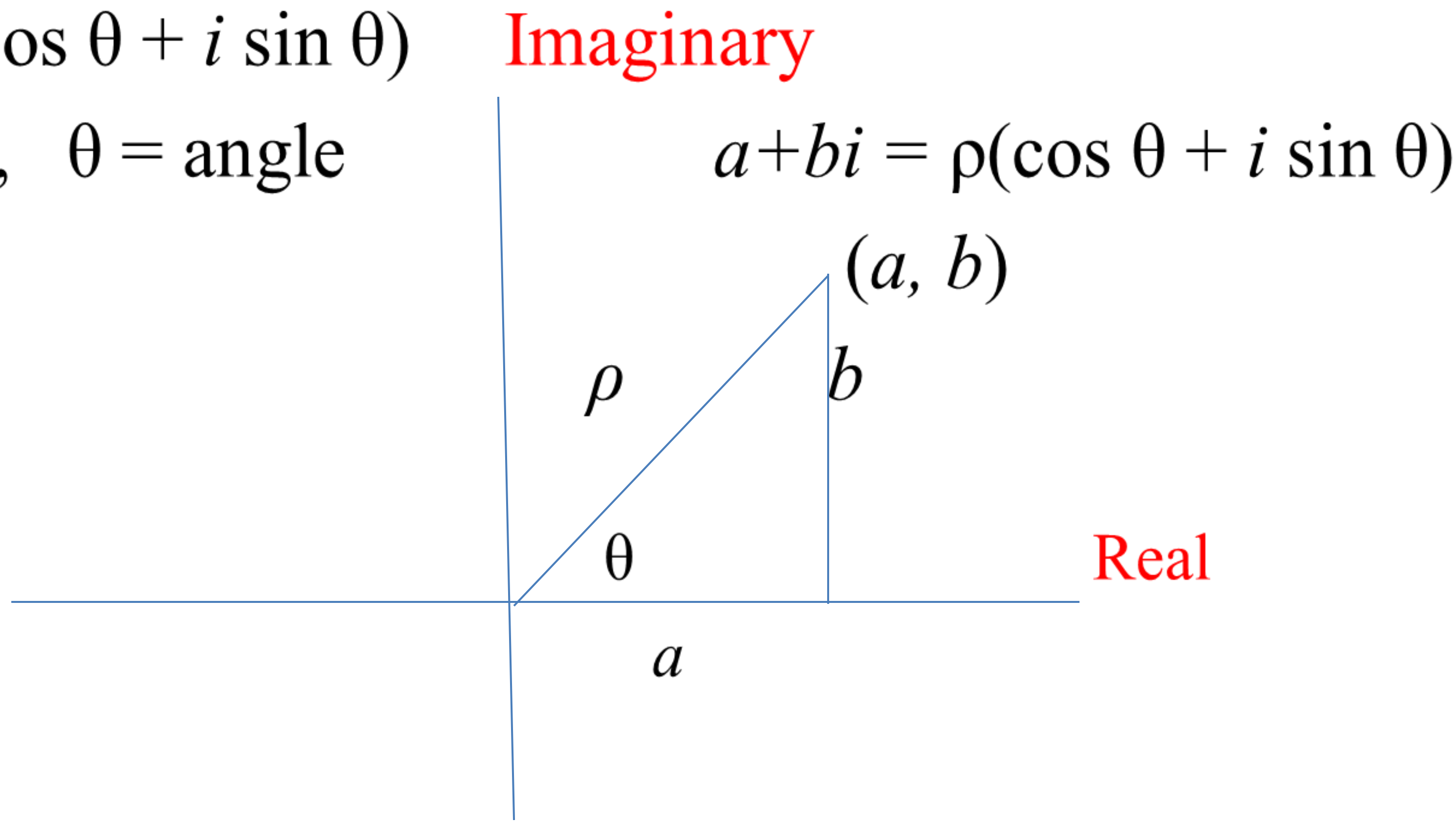


We can also use "radius" and "angle" to represent complex numbers graphically (in **Polar Form**).

Radius = $\rho = \sqrt{a^2 + b^2}$. Using simple trigonometry notations, **$\cos\theta = a/\rho$** and **$\sin\theta = b/\rho$**. So, we can write

$a+bi = \rho(\cos\theta + i\sin\theta)$

ρ = radius, θ = angle



One time around the circle is $360°$ or 2π in radians

The circumference of a unit circle is 2π

So	2π	~	$360°$
	π	~	$180°$
	$\pi/2$	~	$90°$
	0	~	$0°$
	$\pi/4$	~	$45°$
	$\pi/6$	~	$30°$
	$\pi/3$	~	$60°$
	$3\pi/2$	~	$270°$

So, $0, 2\pi, 4\pi, 6\pi, -2\pi, \ldots$ *and* $0°, 360°, 720°, -360°$ are all equivalent to 0 or $0°$

What exactly is $\rho(\cos\theta + i\sin\theta)$?

$1 = \cos(0) + i\sin(0)$ 0°

$i = \cos(\pi/2) + i\sin(\pi/2)$ 90°

$-1 = \cos(\pi) + i\sin(\pi)$ 180°

$-i = \cos(3\pi/2) + i\sin(3\pi/2)$ 270°

$1+i = \sqrt{2}[\cos(\pi/4) + i\sin(\pi/4)]$ 45°

$1+\sqrt{3}i = 2[\cos(\pi/3) + i\sin(\pi/3)]$ 60°

$-\sqrt{3}-i = 2[\cos(7\pi/6) + i\sin(7\pi/6)]$

210°

$$(\cos\theta + i\sin\theta)^2 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$
$$= (\cos^2\theta - \sin^2\theta) + i\,(2\cos(\theta)\sin(\theta))$$
$$= \cos(2\theta) + i\sin(2\theta)$$

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof by Induction

$$(\cos\theta + i\sin\theta)^{n+1} = [\cos(n\theta) + i\sin(n\theta)](\cos\theta + i\sin\theta)$$
$$= [\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta] + i[\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta]$$
$$= \cos((n+1)\theta) + i\sin((n+1)\theta)$$

So,

$$(a + bi)^n = [\rho(\cos \theta + i \sin \theta)]^n$$

$$= \rho^n[\cos(n\theta) + i \sin (n\theta)]$$

Or $(\rho, \theta)^n = (\rho^n, n\theta)$

But there is an easier way to do this.

So,

$$(a + bi)^n = [\rho(\cos\theta + i\sin\theta)]^n$$
$$= \rho^n[\cos(n\theta) + i\sin(n\theta)]$$

Or $(\rho, \theta)^n = (\rho^n, n\theta)$

But there is an easier way to do this by using **exponential function** instead of **trigonometric functions**.

Note that:

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

Why e?

e = Euler's number named after Euler.

e = 2.71828 18284 59045 23536 02874 71352 …

e = Irrational

e = Transcendental (not algebraic)

Interest Rate Problem:

You have \$100 putting into a bank's saving account which pays you 5% annual simple interest. How much have you gotten from the bank (including your original \$100) at the end of 50 years?

$\$100 + \$100 \times 5\% \times 50 = \$350$ (Principal + Interest)

Interest Rate Problem:

You have $100 putting into a bank's saving account which pays you 5% interest compounded annually. How much have you gotten from the bank (including your original $100) at the end of 2 years?

$$(\$100 + \$100 \times 5\%) + [(\$100 + \$100 \times 5\%) \times 5\%]$$
$$= \$100 \times [(1+5\%) + (1+5\%) \times 5\%]$$
$$= \$100 \times (1+5\%) \times (1+5\%)$$
$$= \$100(1+5\%)^2 = \$110.25 \text{ (Principal + Interest)}$$

Interest Rate Problem:

You have $100 putting into a bank’s saving account which pays you 5% interest compounded annually. How much have you gotten from the bank (including your original $100) at the end of 50 years?

$$\$100(1+5\%)^{50} = \$1{,}146.74 \quad \text{(Principal + Interest)}$$

Interest Rate Problem:

You have \$100 putting into a bank's saving account which pays you 5% interest compounded monthly. How much have you gotten from the bank (including your original \$100) at the end of 50 years?

$$\$100\left(1+\frac{5\%}{12}\right)^{12\times 50} = \$1,211.94 \quad \text{(Principal + Interest)}$$

<u>Interest Rate Problem:</u>

You have $100 putting into a bank's saving account which pays you 5% interest compounded daily. How much have you gotten from the bank (including your original $100) at the end of 50 years?

$$\$100(1+\frac{5\%}{365})^{365\times 50} = \$1{,}218.04 \quad \text{(Principal + Interest)}$$

Interest Rate Problem:

In general, you have $100 putting into a bank's saving account which pays you 5% interest compounded n times in a year. How much have you gotten from the bank (including your original $100) at the end of 50 years?

$$\$100\left(1+\frac{5\%}{n}\right)^{n\times 50} \quad \text{(Principal + Interest)}$$

Interest Rate Problem:

In general, you have $100 putting into a bank's saving account which pays you 5% interest compounded secondly (n = 365 × 24 × 60 × 60 = 31536000). How much have you gotten from the bank (including your original $100) at the end of 50 years?

$$\$100\left(1+\frac{5\%}{31536000}\right)^{31536000\times 50} = \$1227.61$$

This amount does not differ that much from the annual compounding amount of $1218.04.

Interest Rate Problem:

$$P_0(1+\frac{r}{n})^{nt} \quad (P_0 = \text{beginning principal}$$

$$r = \text{interest rate, } t = \text{time})$$

When $n \rightarrow \infty$, the compounding is called "compounded continuously."

$$P_0(1+\frac{r}{n})^{nt} = P_0(1+1/(\frac{n}{r}))^{(n/r)rt} = P_0((1+1/m)^m)^{rt}$$

where $m = n/r$.

Interest Rate Problem:

$$P_0(1+\frac{r}{n})^{nt} = P_0(1+1/(\frac{n}{r}))^{(n/r)rt} = P_0((1+1/m)^m)^{rt}$$

$m=n/r$ When $n \rightarrow \infty$, $m \rightarrow \infty$.

The question becomes what is $(1+1/m)^m$ when $m \rightarrow \infty$?

<u>Interest Rate Problem:</u>

m		$(1+1/m)^m$
1		2
10		2.59374246…
100		2.70481383…
1000		2.716923933…
10000		2.718145936…
100000		2.718268303…
1000000		2.718281378…
∞		e = 2.718281828…

Because e appeared naturally on this and many other applications, e is called the "***natural base***."

Euler's Formula

$$a+bi = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$$

$$(a+bi)^n = \rho^n(\cos n\theta + i\sin n\theta)$$

$$(a+bi)^n = (\rho e^{i\theta})^n = \rho^n e^{in\theta}$$

Euler's Formula $a+bi = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$

Proof(1). Taylor's Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Euler's Formula $a+bi = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$

Proof(1). Taylor's Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Since

$i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1$

Replace x with $i\theta$.

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{or}$$

$$\boldsymbol{\rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}}$$

Euler's Formula $a+bi = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$

Proof(2). Consider the function $f(\theta) = e^{-i\theta}(\cos(\theta)+i\sin(\theta))$.

Take derivative of $f(\theta)$ using Product Rule:

$$f'(\theta) = -ie^{-i\theta}[(\cos(\theta)+i\sin(\theta)] + e^{-i\theta}[-\sin(\theta)+i\cos(\theta)] = 0$$

So, $f(\theta) = \text{Constant}$ and since $f(0) = 1$, $f(\theta) = 1$ or

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textit{or}$$

$$\boldsymbol{\rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}}$$

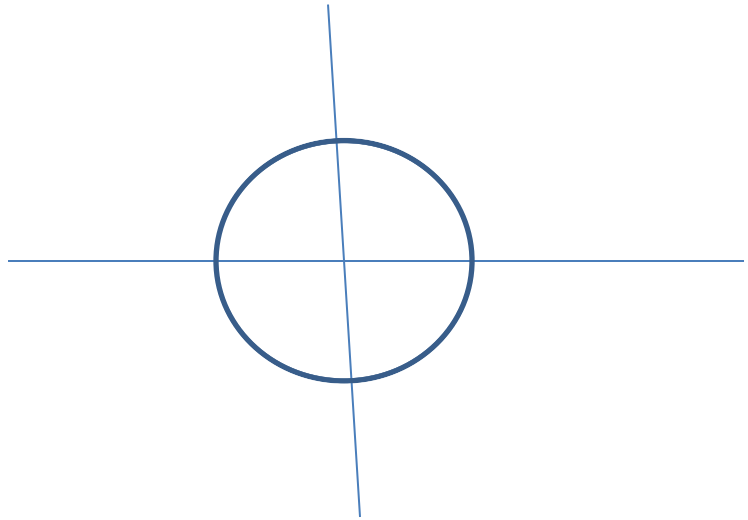
Applications of Euler's Formula

$$a+bi = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$$

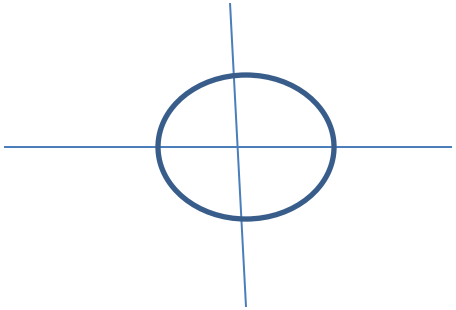
Let $\theta = -\pi$. $(-180°)$ Then

$e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = -1.$

So $$\mathbf{e^{-\pi i} + 1 = 0}.$$



$$a+bi = \rho e^{i\theta}$$



Application:

$$e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1 = \cos(2\pi n) + i\sin(2\pi n) = e^{(2\pi n i)}$$

$$[\rho(a+bi)]^{1/2} = \rho^{1/2}e^{i\theta/2} \text{ or } \rho^{1/2}e^{i(\theta+2\pi k)/2} = \rho^{1/2}e^{i(\theta/2+2\pi k/2)} \quad \text{where } k = 0, 1.$$

$$[\rho(a+bi)]^{1/3} = \rho^{1/3}e^{i\theta/3} \text{ or } \rho^{1/3}e^{i(\theta+2\pi k)/3} = \rho^{1/3}e^{i(\theta/3+2\pi k/3)} \quad \text{where } k = 0, 1, 2.$$

$$[\rho(a+bi)]^{1/5} = \rho^{1/5}e^{i\theta/5} \text{ or } \rho^{1/5}e^{i(\theta+2\pi k)/5} = \rho^{1/5}e^{i(\theta/5+2\pi k/5)} \quad \text{where } k = 0, 1, 2, 3, 4.$$

In general,

$$\mathbf{[\rho(a+bi)]^{1/n} = \rho^{1/n}e^{i(\theta/n+2\pi k/n)} \quad \text{where } k = 0, 1, 2, \ldots, n-1.}$$

$$[\rho(a+bi)]^{1/n} = \rho^{1/n} e^{i(\theta/n+2\pi k/n)}$$

where $k = 0, 1, 2, \ldots, n-1$.

So, $a+bi$ has n number of n^{th} roots and these n^{th} roots divide the circle into n equal parts and all we have to know is the first n^{th} root and then keep adding those equal parts.

For example, there are 3 answers to cube roots of i.

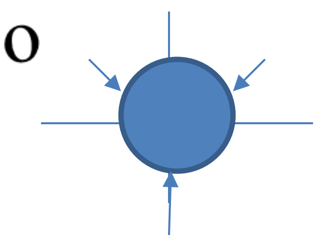
i is located at angle $\pi/2$ and the radius is $\rho = 1$.

$$i = e^{i(\pi/2+2n\pi)} \quad n = 0, 1, 2, 3, \ldots$$

Therefore, $\sqrt[3]{i} = i^{1/3} = e^{i(\pi/6+2n\pi/3)} \quad n = 0, 1, 2, 3, \ldots$

However, when $n = 3$, $2n\pi/3 = 2\pi$ and

$\pi/6 + 2\pi$ and $\pi/6$ represent the same angle so

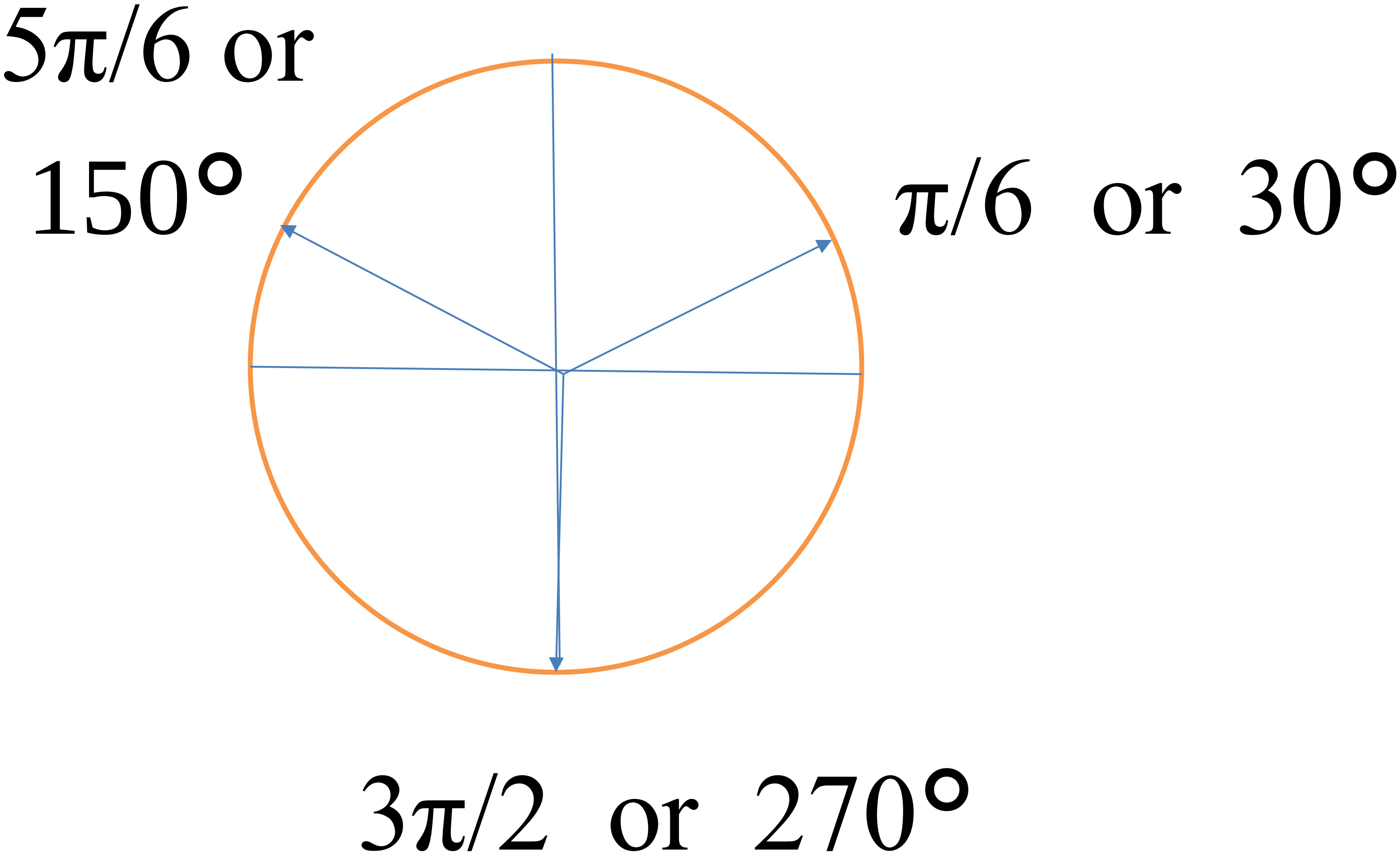


$\sqrt[3]{i} = i^{1/3} = e^{i(\pi/6+2n\pi/3)} \quad n = 0, 1, 2$ or

$$\sqrt[3]{i} = e^{i(\pi/6)}, \quad e^{i(5\pi/6)}, \quad e^{i(9\pi/6)} = e^{i(3\pi/2)}$$

These represent the 3 points on the unit circle at angles $30°, 150°, 270°$ each separated by $360/3 = 120°$.

$$i^{1/3} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad -i$$


5π/6 or
150°
π/6 or 30°
3π/2 or 270°

Research Problem

$2^3 = 2\times2\times2$

$2^A = ????$ where $A = \begin{bmatrix} 3 & 2 \\ 1 & 8 \end{bmatrix}$

What is the best way to define exponent using matrices? It turns out very useful in physics.

Let us look at e^A.

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$$

$$e^A = I + A + A^2/2! + A^3/3! + A^4/4! + \ldots$$

Let $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $A^2 = A^3 = A^4 = I$.

$$e^I = I + I + I/2! + I/3! + \ldots = eI = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

What about if $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$?

What about if $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$?

$e^A = \mathrm{I} + A + A^2/2! + A^3/3! + A^4/4! + \ldots$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 \\ 0 & 2^3 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 2^4 \end{bmatrix}$

$e^A = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$

In general, if one can "diagonalize" A, that is, if one can find a matrix U so that $A = UDU^{-1}$ where D is a diagonal matrix, then $e^A = Ue^DU^{-1}$.

Take $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. Then $A = UDU^{-1} =$ where

$U = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $U^{-1} = (-1/4)\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix}$.

$e^A = U\begin{bmatrix} 1/e & 0 \\ 0 & e^3 \end{bmatrix}U^{-1}$

Problems:

1. $x^2 + 2x + 5 = 0$

$$x = -1 \pm 2i = \rho e^{i\theta} = \rho(\cos(\theta) + i\sin(\theta))$$

$$= \sqrt{5}\left(-\frac{1}{\sqrt{5}} \pm \frac{2}{\sqrt{5}}i\right)$$

where $\rho = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

and $\cos(\theta) = -\frac{1}{\sqrt{5}}$ and $\sin(\theta) = \pm\frac{2}{\sqrt{5}}$

or $\theta \approx 2.034^{R}$ or $116.565°$

$\theta \approx 4.244^{R}$ or $143.435°$

So, $x = \sqrt{5}e^{2.034i}$ or $\sqrt{5}e^{4.244i}$

Problems:

2. $x^2 - 1 = 0$

$$x = 1 = \rho e^{i\theta} = \rho(\cos(\theta)+i\sin(\theta))$$

$$= 1(1 + 0i) \qquad \text{where} \quad \rho = 1$$

$$\text{and} \quad \cos(\theta) = 1 \quad \text{and} \quad \sin(\theta) = 0$$

$$\text{or} \qquad \theta = 0 \quad \text{or} \quad 0°$$

So, $x = e^{i(0+2n\pi/2)} = (\cos(0+2n\pi/2)+i\sin(0+2n\pi/2))$

where $n = 0, 1$

$$= \; e^{i(0)} \; \textbf{(0°)} \qquad \text{or} \quad e^{i(\pi)} \; \textbf{(180°)}$$

$$= \; 1 \qquad\qquad \text{or} \quad -1$$

Problem:

3\. $x^3 - 1 = 0$

$$x = 1 = \rho e^{i\theta} = \rho(\cos(\theta)+i\sin(\theta))$$

$$= 1(1 + 0i) \qquad \text{where} \quad \rho = 1$$

and $\quad \cos(\theta) = 1$ and $\sin(\theta) = 0$

or $\quad \theta = 0$ or $0°$

So, $x = e^{i(0+2n\pi/3)} = (\cos(0+2n\pi/3)+i\sin(0+2n\pi/3))$

where $n = 0, 1, 2$

$\cos(0)+i\sin(0) = 1$ **(0° or 360°)**

$\cos(2\pi/3)+i\sin(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **(120°)**

$\cos(4\pi/3)+i\sin(4\pi/3) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **(240°)**

Problem:

4. $x^4 - 1 = 0$

$$x = 1 = \rho e^{i\theta} = \rho(\cos(\theta) + i\sin(\theta))$$

$$= 1(1 + 0i) \quad \text{where} \quad \rho = 1$$

and $\cos(\theta) = 1$ and $\sin(\theta) = 0$

or $\theta = 0$ or $0°$

So, $x = e^{i(0+2n\pi/4)} = (\cos(0+n\pi/2) + i\sin(0+n\pi/2))$

where $n = 0, 1, 2, 3$

$\cos(0) + i\sin(0) = 1 = e^{0i} = 1$ **(0°)**

$\cos(\pi/2) + i\sin(\pi/2) = i = e^{\frac{\pi}{2}i} = e^{1.57\ldots i}$ **(90°)**

$\cos(\pi) + i\sin(\pi) = -1 = e^{\pi i} = e^{3.14159\ldots i}$ **(180°)**

$\cos(3\pi/2) + i\sin(3\pi/2) = -i = e^{\frac{3\pi}{2}i} = e^{4.712\ldots i}$ **(270°)**

Problems:

5. $x^2 = 9$ $\qquad x = \pm 3$
6. $2^x = 8$ $\qquad x = 3$
7. $x^i = 1$ $\qquad x = ???$
8. $x^i = 2$ $\qquad x = ???$
9. $i^x = 1$ $\qquad x = 0\ ???$

Two Important Points:

1. Numbers must be in the form of $a+bi$.
2. Must be in base e.

7. $x^i = 1$

$$x^i = 1 = e^{i(0+2m\pi)} \qquad m = \text{integers}$$

$$x = (x^i)^{1/i} = \left(e^{i(2m\pi)}\right)^{1/i} \qquad m = \text{integers}$$

$$x = e^{2m\pi} \qquad m = \text{integers}$$

Multiple answers.

$m = 0$	$x = e^{2m\pi} = e^{0} = 1$	**$(1)^i$**	**$= 1$**
$m = 1$	$x = e^{2m\pi} = e^{2\pi} \approx 535.49$	**$(535.49..)^i$**	**$= 1$**
$m = -1$	$x = e^{2m\pi} = e^{-2\pi} \approx 0.001867\ldots$	**$(0.001867\ldots)^i$**	**$= 1$**
$m = 2$	$x = e^{2m\pi} = e^{4\pi} \approx 286751.3131\ldots$	**$(28675.3131\ldots)^i$**	**$= 1$**

8. $x^i = 2$

$x^i = 2 = 2e^{i(2m\pi)}$ $\quad m$ = integers

$x = (x^i)^{1/i} = (2e^{i(2m\pi)})^{1/i}$ $\quad m$ = integers

$x = 2^{1/i}\, e^{2m\pi}$ $\quad m$ = integers

We are back to trying to find $2^{1/i}$

??????????????

Log or *ln* and *e* functions are inverses of each other.

$ln(x) = y$ means $e^y = x$

So "*ln*" always represents the "power"

$ln(e^x) = x$ $e^{ln(x)} = x$

$ln(pq) = ln(p) + ln(q)$ $e^m \times e^n = e^{m+n}$

$ln(p/q) = ln(p) - ln(q)$ $e^m \div e^n = e^{m-n}$

$ln(p^n) = n ln(p)$ $(e^m)^n = e^{mn}$

8. $x^i = 2$

$$x = 2^{\frac{1}{i}} = 2^{\frac{1\times i}{i\times i}} = 2^{-i}$$

$$= e^{ln(2^{-i})} = e^{-iln(2)} \quad (ln \text{ is the inverse of } e)$$

$$= e^{-ln(2)i} = e^{i\theta} \quad \text{where} \quad \theta = -ln(2).$$

Check: $x^i = (e^{-ln(2)i})^i = e^{-ln(2)(-1)} = e^{ln(2)} = 2.$

So, $x = \cos(-ln(2)) + i\sin(-ln(2))$

$= \cos(-0.693147\ldots) + i\sin(-0.693147\ldots)$

$= \cos(-39.7\ldots°) + i\sin(-39.7\ldots°)$

$= \mathbf{(0.769239\ldots) - (0.63896\ldots)}\boldsymbol{i}$

9. $i^x = 1$ Take ln on both sides.

$x \times \ln(i) = \ln(1)$ $x \times \ln(e^{i(\boldsymbol{\pi}/2+2n\pi)}) = \ln(e^{i(0+2m\pi)})$

$x = \ln(e^{i(0+2m\pi)})/\ln(e^{i(\boldsymbol{\pi}/2+2n\pi)}) = i(0+2m\boldsymbol{\pi})/i(\boldsymbol{\pi}/2+2n\boldsymbol{\pi})$

$= 2m/(1/2+2n) = \mathbf{4m/(1+4n)}$

where m =integers and n = integers

Check: i^$\mathbf{4m/(1+4n)} = (i^4)$^ $\mathbf{m/(1+4n)}$ = 1^ $\mathbf{m/(1+4n)}$ = 1

Multiple Answers

$m = 0, n = 1$:	$x = 0$	$i^0 = 1$
$m = 1, n = 1$:	$x = 4/5$	$i^{(4/5)} = (i^4)^{1/5} = 1^{1/5} = 1$
$m = 2, n = 10$:	$x = 8/41$	$i^{(8/41)} = (i^8)^{1/41} = 1^{1/41} = 1$

Back to

$$1^x = 2$$

9. $1^x = 2$

$$ln(1^x) = ln(2)$$

$$x\,ln(1) = ln(2) \quad \text{or} \quad x = \frac{ln(2)}{ln(1)}$$

$ln(1) = 0$ in real numbers so $x = \dfrac{ln(2)}{ln(1)}$ is undefined.

But, in complex numbers,

$$ln(1) = ln(e^{2\pi ni}) \qquad n = \text{integers}$$

$$= 2\pi ni$$

$$1^x = 2$$

$$x = \frac{\ln(2)}{\ln(1)} = \frac{\ln(2)}{\ln(e^{2\pi n i})}$$

where $n \neq 0$

$$1^x = 2$$

$$x = \frac{\ln(2)}{\ln(1)} = \frac{\ln(2)}{\ln(e^{2\pi n i})} \qquad \text{where } n \neq 0$$

$$= \frac{\ln(2)}{2\pi n i} = -i\left(\frac{\ln(2)}{2\pi n}\right) = -\frac{0.11}{n}\,i \qquad \text{where } n \neq 0$$

Check: **$n = 1$** $\quad x = -i\left(\frac{\ln(2)}{2\pi}\right) = -0.11i$ and

$$1^x = 1^{-i\left(\frac{\ln(2)}{2\pi}\right)} = \left(e^{2\pi i}\right)^{-i\left(\frac{\ln(2)}{2\pi}\right)} = e^{\ln(2)} = 2.$$

For general n: $\quad x = -i\left(\frac{\ln(2)}{2\pi n}\right) = -\frac{0.11}{n}i \quad (n = \ldots, -2, -1, 1, 2, \ldots)$

and $\quad 1^x = 1^{-i\left(\frac{\ln(2)}{2\pi n}\right)} = \left(e^{2n\pi i}\right)^{-i\left(\frac{\ln(2)}{2\pi n}\right)} = e^{\ln(2)} = 2.$

Riemann Hypothesis

Question:
How many prime numbers are there?

•

There are ***infinitely*** many prime numbers.

Take

$2+1 = 3$ (new prime)

$2\times3+1 = 7$ (new prime)

$2\times3\times5+1 = 31$ (new prime)

$2\times3\times5\times7+1 = 211$ (new prime)

$2\times3\times5\times7\times11+1 = 2311$ (new prime)

Take

$2+1 = 3$ (new prime)

$2\times3+1 = 7$ (new prime)

$2\times3\times5+1 = 31$ (new prime)

$2\times3\times5\times7+1 = 211$ (new prime)

$2\times3\times5\times7\times11+1 = 2311$ (new prime)

$2\times3\times5\times7\times11\times13+1 = 30031 = 59\times509$

Suppose we have only a finite number of prime numbers listed as $p_1, p_2, p_3, p_4, \ldots, p_n$.

Define a new number

$$\boldsymbol{q = p_1 \times p_2 \times p_3 \times p_4 \times \ldots \times p_n + 1}.$$

(1) ***q*** is larger than any of our prime numbers.

(2) None of our original list of *n* prime numbers can be a factor of ***q*.**

So, either ***q*** is a prime number in which case we have found a new prime number that is larger than all prime numbers on our original list or ***q*** has a prime factor that is not on our list. Either way it contradicts our original assumption that we have only a finite number of prime numbers.

Instead of trying to find a precise formula producing all prime numbers, mathematicians in the late 19th century tried to find an estimate as to how frequent prime numbers appeared.

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How many prime numbers in between 1 and 20?

***8 prime numbers**: 2, 3, 5, 7, 11, 13, 17, 19.*

How many prime numbers in between 1 and 1000?

***That would take a lot more time*.** **(91, 221, 289, 323, 391, 437, 529, 667, 899)**

How many prime numbers in total?

Euclid (father of geometry) proved 2000 years ago that there are infinitely many prime numbers using an ingenious way of creating new prime numbers by adding 1 to the product of all prime numbers.

Suppose 2 and 3 are the only primes. Then $2 \times 3 + 1 = 7$ is a prime.

Suppose 2, 3, and 5 are the only primes. Then $2 \times 3 \times 5 + 1 = 31$ is a prime.

Suppose 2, 3, 5, and 7 are the only primes. Then $2 \times 3 \times 5 \times 7 + 1 = 211$ is a prime.

Suppose 3, 5, and 7 are the only primes. Then $3 \times 5 \times 7 + 1 = 106$. Although 106 is **not** a prime, there should be at least one missing prime factor in 106 since none of 3, 5, or 7 is a factor. And, of course, missing prime factors are 2 and 53.

So, by adding 1 to the product of all the prime numbers either we have created a new prime number or if that product plus 1 is not prime, then it must have a prime factor that is not on our list of ALL prime numbers.

In 1896, a French mathematician Hadamard proposed that the number of prime numbers less than or equal to n, we called it $\pi(n)$, is approximately equal to $n/\ln(n)$. In another word, $\pi(n)$ is getting closer and closer to $n/\ln(n)$ as n gets to be larger and larger.

$$\pi(n) \approx \frac{n}{\ln(n)}$$

This is better known as **Prime Number Theorem (PNT)**

.

For examples:

There are **6** prime numbers ≤ 16 (2, 3, 5, 7, 11, 13) and 16/ln(16) = **5.77**….

There are **11** prime numbers ≤ 36 (2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31) and 36/ln(36) = **10.0599**….

Note that $\pi(n) \approx \frac{n}{\ln(n)}$ is not a precise formula.

The larger the number *n*, the better the estimate.

For example:

There are approximately $1000000/\ln(1000000) \approx 72{,}382$ prime numbers ≤ 1000000

Is there a pattern to the appearance of prime numbers?

Unfortunately, after thousands of years of trying, none of our most brilliant mathematicians had found a way to guarantee in finding a new prime.

Riemann Hypothesis provides us with the most promising answer but no one had found a way to prove or disprove that 150 years old hypothesis. Many mathematicians consider this hypothesis to be one of the most important questions in mathematics.

x	$\pi(x)$	$\pi(x) - x/\ln x$	$\pi(x) \div x / \ln x$
10	4	−0.3	0.921
10^2	25	3.3	1.151
10^3	168	23	1.161
10^4	1229	143	1.132
10^5	9592	906	1.104
10^6	78498	6116	1.084
10^7	664579	44158	1.071
10^8	5761455	332774	1.061
10^9	50847534	2592592	1.054
10^{10}	455052511	20758029	1.048
10^{11}	4118054813	169923159	1.043
10^{12}	37607912018	1416705193	1.039
10^{13}	346065536839	11992858452	1.034
10^{14}	3204941750802	102838308636	1.033
10^{15}	29844570422669	891604962452	1.031

The proof of this theorem is extremely complicated. In 1980, an American mathematician Newman found a “short” proof to this theorem but it requires some very advanced mathematics techniques such as Cauchy’s Integral Theorem from Complex Analysis..

Riemann Hypothesis was formulated by Bernhard Riemann in 1859 attempting to shine some light to one of the oldest questions in mathematics. What is the pattern of the prime numbers among all counting numbers?

A proof of the **Riemann Hypothesis** would add to pure understanding of the prime numbers and the way they are distributed. Besides having implications in mathematics well beyond the pattern of primes, it would have ramifications in physics and modern communications technology.

In order to better understand the history and meaning of Riemann Hypothesis, one needs to have a better knowledge in *complex numbers* and *infinite series*.

In 1740, Euler introduced a

zeta function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

where s is real number

From Calculus,

$s = 1$ Divergent Harmonic Series

$s = 2$ Converges to $\frac{\pi^2}{6}$

$s = 4$ Converges to $\frac{\pi^4}{90}$

In general, this *zeta function* is divergent if $s \leq 1$ and

convergent if $s > 1$

(s = real numbers).

What does this ***zeta function*** have to do with prime numbers?

Euler proved that for any real number $s > 1$, this ***zeta function*** is equal to the infinite product

$$\frac{1}{1-(\frac{1}{2})^s} \times \frac{1}{1-(\frac{1}{3})^s} \times \frac{1}{1-(\frac{1}{5})^s} \times \frac{1}{1-(\frac{1}{7})^s} \times \ldots \times \frac{1}{1-(\frac{1}{p})^s} \times \ldots$$

p = prime number.

Although this ***zeta function*** is linked to prime numbers somewhat, as a function, it is from real numbers to real numbers. It is a one–dimensional object. It does not have sufficient geometric structure to help uncover the pattern of the primes. To do that, we need to move up to two–dimensions. Riemann was the first person who made this switch using complex numbers. So, this new complex number zeta function is referred to as *Riemann's zeta function*.

The analytic continuation of *zeta function* can be taken in many forms.

Define the Gamma Function Γ(z) that is related to the operation !

$$\Gamma(n) = (n-1)! \quad \text{and} \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \mathbf{R}(z) \geq 1$$

It turns out

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\, \Gamma(1-z)\, \zeta(1-z)$$

In his 1859 paper, Riemann observed that for each negative **even** integers –2, –4, –6, …

zeta function is zero and he considered these zeros the "trivial zeros." He then showed that *zeta function* has infinitely many other complex zeros besides these real zeros.

He then conjectured that all those other zeros have the form of $z = \frac{1}{2} + bi$.

One of the analytic continuation ***zeta function*** formulas tells us that the values of the ***zeta function*** at *z* are closely related to its values at 1–*z* which means the ***zeta function*** has a certain symmetry about the critical line. Maybe Riemann thought that this symmetry forces all zeros to be on the line of symmetry which is the **vertical line (called the *critical line*) through the point ½** on the real axis.

FACTS about *Riemann Zeta Function*:

1. It has trivial zeros at z = negative even integers.
2. It has infinitely many complex zeros.
3. Using computers, mathematicians have managed to show that Riemann Hypothesis is true for the first 1.5 billion zeros (very close to the real axis.)
4. Most mathematicians believe RH is true.

FACTS about *Riemann Zeta Function*:

5. In 1972, Hugh Montgomery found a formula that describes the spacing between the zeros along the critical line.
6. Physicists immediately recognized that formula would give the spacing between the energy levels of a quantum chaotic system.
7. It is known that any non–trivial zero lies in the open strip $\{s \in \mathbf{C}: 0 < \mathrm{Re}(s) < 1\}$, which is called the ***critical strip***.
8. In 1914, Godfrey Hardy proved that $\zeta(1/2+it)$ has infinitely many real zeros..

Exercise (Advanced)

Solve the following equations for $\boldsymbol{x}$ in Complex Numbers (a+bi) or (r, θ) or $re^{i\theta}$:

1. $x^5 - 1 = 0$
2. $x^2 - i = 0$
3. $x^2 - (3+4i) = 0$
4. $x^4 + 1 = 0$
5. $2x^5 + 1 = 0$
6. $x^6 + 64 = 0$
7. $x = i^i$
8. $x^i - i = 0$
9. $1^x = 2$
10. $i^x = 1$
11. $x^3 + i = 0$
12. $1^x = -1$

- (BONUS) Define "odd" and "even" Gaussian Integers (complex numbers of integer parts). Prove that they satisfy the properties of being "odd" or "even" as stated in lecture.

2025 Berkeley International Math Summer Camp

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July 27 (Sunday) – Arrive Los Angeles
July 28, 29, 30, 31 – Los Angles
- Tour UCLA, USC, Caltech campuses
- Universal Studio
- Outlet Shopping
- Museum Visits
- City Sightseeing
- San Diego Math Circle

August 1 (Friday) – Bus drives to Berkeley
- Visit Sacramento Capital
- Visit College Campus
- Check into Berkeley Unit I dormitory

August 2 (Saturday) – San Francisco Area Tour, Stanford Campus Tour
August 3, 4, 5, 6, 7, 8 – Berkeley
- Lectures
- Preliminary WMTC Math Competition
- Berkeley Campus Tour

August 9 (Saturday) – Depart from San Francisco Airport