# BERKELEY MATH CIRCLE

## Problem Solving Techniques:

## Solutions,
## Not Answers

**Instructor: Patricio Angulo**

# Solutions, NOT Answers Math Contest

- Choose Teams of 6-8 classmates
- Come up with a team name (Be creative!)
- Pick a team captain
- Your team should consist of classmates that excel in at least one of the following (diversity of thought is key!):

    - Algebra
    - Geometry
    - Problem Solving/Proofs
    - Writing clear and concise **Solutions!**

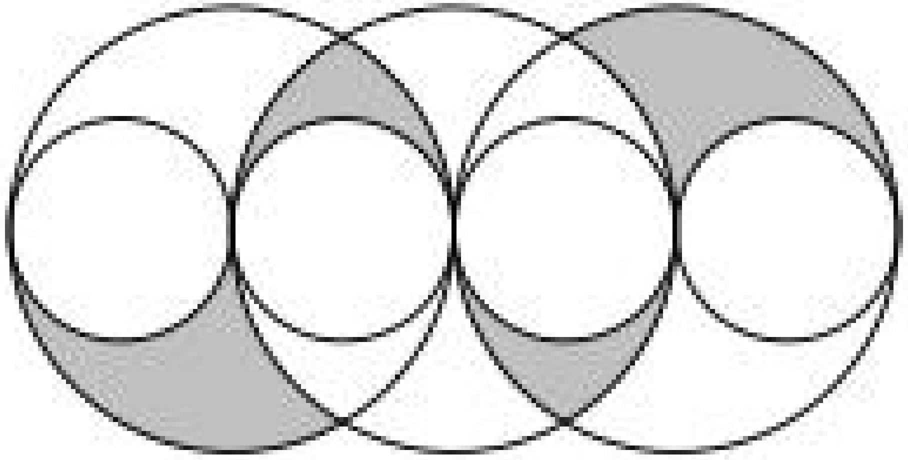
- All group members must sign off on the solution, and only Co-Co-Clear 10's will be accepted!
- You will have 15 minutes/question. Two Q's have a bonus question! The Bonus part of the question can only be obtained **AFTER** receiving a Co-Co-Clear 10 on the original question!
- Points awarded for order in which correct answer is turned in
- All teams get the next question at the same time (only **two** Q's out at the same time)
- Once we get to the $3^{rd}$ and $4^{th}$ question, a team must decide to move on from previous questions
- We will end at 745pm to discuss results and award prizes
- Prizes awarded to the top three teams!
- Good Luck Mathing!

# Solutions, NOT Answers Math Contest

## Timing

- 630pm – 645pm: Teams/Names/Rules
- 645pm – 7pm: Q1
- 700pm – 715pm: Q2
- 715pm – 730pm: Q3
- 730pm – 745pm: Q4
- 745pm: Discuss Contest; Show Solutions
- 750pm: Awards & Recognition
- 755pm: Clean-up!

The diagram shows three large circles of equal radius, and four small circles of equal radius. The centers of all circles, and all points of contact between any of the circles, lie on one straight line. The radius of each small circle is 1. What is the total area of the shaded regions?



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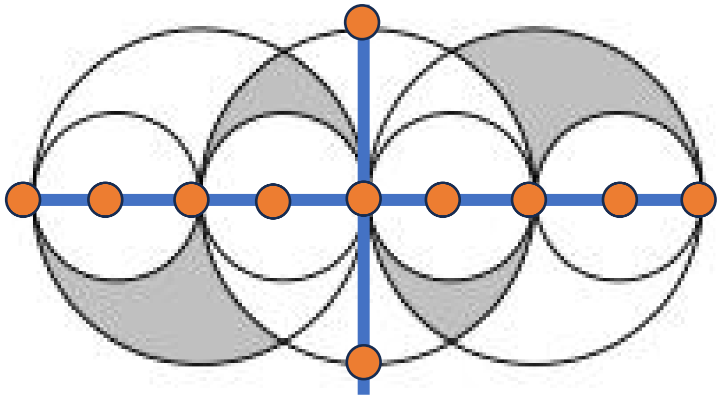


Since “the centers of all circles, and all points of contact between any of the circles, lie on one straight line”, we can draw **lines of symmetry** as shown. The lines of symmetry allow us to conclude that the two small circles are **perfectly inscribed** into each big circle, thus allowing us to conclude that the radius of each big circle is 2cm.

Additionally, we can **reflect** the two outside circles over the vertical line shown, thereby creating one circle wherein the only spaces not shaded are the two small inscribed circles (see the spider-man “venom” like picture below).

Therefore, the amount of the shaded area:

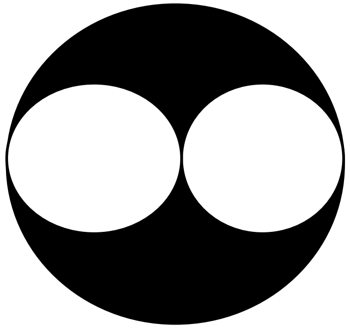
Shaded Area = Area of big circle – 2(Area of small circle)

Area of any circle → $A = (\pi)(r^2)$

Shaded Area = $(\pi)(r\,(\text{big})^2) - 2[(\pi)(r(\text{small})^2)]$

Shaded Area = $(\pi)(2^2) - 2(\pi)(1)^2 \rightarrow 4\pi - 2\pi = 2\pi$

**The total area of the shaded regions is 2π.**



Pat and Mat were trying to calculate the average of two numbers $a$ and $b$ using their calculator. First, Pat took the calculator and typed in a + b ÷ 2 and obtained 30. Then Mat took the same calculator and typed in b + a ÷ 2 and obtained 18. What was the correct average of the two numbers?

**SOLUTION**

Pat's Calculator Input: $a + b \div 2 = 30$
Due to the order of operations, the calculator solved Pat's input as:
$a + (b \div 2) = 30 \rightarrow a + b/2 = 30 \rightarrow (2a + b)/2 = 30 \rightarrow$ **$2a + b = 60$**

Mat's Calculator Input: : $b + a \div 2 = 18$.
Due to the order of operations, the calculator solved Mat's input as:
$b + (a \div 2) = 18 \rightarrow b + a/2 = 18 \rightarrow (2b + a)/2 = 18 \rightarrow$ **$2b + a = 36$**

Adding the bolded equations, we have:

$$\begin{array}{r} 2a + b = 60 \\ \underline{+\ 2b + a = 36} \\ 3a + 3b = 96 \\ a + b = 32 \end{array}$$

Since $a + b = 32$, the average of $a + b$ is $32/2 = 16$

**The correct average of the two numbers is 16.**

***Bonus Question! What are the values of a and b individually?***

**SOLUTION**

***Bonus Question! What are the values of a and b individually?***

From the previous solution, we have:

$a + b = 32$
$a = 32 – b$

Also, using Pat's equation and then substituting:

$a + (b \div 2) = 30$
$(32 – b) + b/2 = 30$
$32 – b/2 = 30$
$b/2 = 2$
$b = 4$

Since $a + b = 32$, $a = 28$

**The values of a and b individually are 28 and 4, respectively.**

What is the greatest whole number less than 1000 that:

(1) can be expressed as the sum of two consecutive whole numbers

AND

(2) can be expressed as the sum of three consecutive whole numbers

AND

(3) can be expressed as the sum of five consecutive whole numbers?

**SOLUTION**

The sum of any two consecutive numbers must be odd:
$n + (n+1) = 2n + 1 \rightarrow 2n$ = even, therefore $2n + 1$ = odd

The sum of any three consecutive numbers must be a multiple of 3:
$n + (n + 1) + (n + 2) = 3n + 3 = 3(n +1) \rightarrow$ the final number is divisible by 3.

The sum of any five consecutive numbers must be a multiple of 5:
$n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10 = 5(n +2) \rightarrow$ the final number is divisible by 5.

The greatest whole number we want is **the greatest odd multiple of both 3 and 5**. Since odd multiples of 5 end in 5, and starting back from 1000, we have:

995: not a multiple of 3, so therefore this does not fit the criteria
985: not a multiple of 3, so therefore this does not fit the criteria
975: an odd multiple of 3 and 5. Yeah!

**The greatest whole number less than 1000 that fits the aforementioned criteria is 975.**

***Bonus Question! What are the values for each specific criteria that still satisfy the overall criteria?***

**SOLUTION**
***Bonus Question! What are the values for each specific criteria that still satisfy the overall criteria?***

The number we found was 975. Looking at each specific criteria:

The sum of two consecutive numbers:
$n + (n+1) = 2n + 1 = 975$
$2n = 974$
$\mathbf{n = 487}$
$\mathbf{n + 1 = 488}$

The sum of three consecutive numbers:
$n + (n + 1) + (n + 2) = 3n + 3 = 975$
$3n = 972$
$\mathbf{n = 324}$
$\mathbf{n + 1 = 325}$
$\mathbf{n + 2 = 326}$

The sum of five consecutive numbers:
$n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10 = 975$
$5n = 965$
$\mathbf{n = 193}$
$\mathbf{n + 1 = 194}$
$\mathbf{n + 2 = 195}$
$\mathbf{n + 3 = 196}$
$\mathbf{n + 4 = 197}$