Berkeley Math Circle, Intermediate I, October 30 2024

Euclidean geometry

Challenge: how would you describe a straight line only using the notion of distance?

- **Problem 1:** What is the sum of the angles in a triangle? In a quadrilateral? In a pentagon? Does it depend on the pentagon?
- **Problem 2:** Draw a line L and a point p not on L. How many lines are there that pass through p and are parallel to L?
- **Problem 3:** What happens to area when we scale? That is, if I have a figure F with area a and scale it by a factor c, what is the area of the scaled figure?
- **Problem 4:** What is the sum of the angles in a triangle in the Euclidean plane? What does that actually mean geometrically?

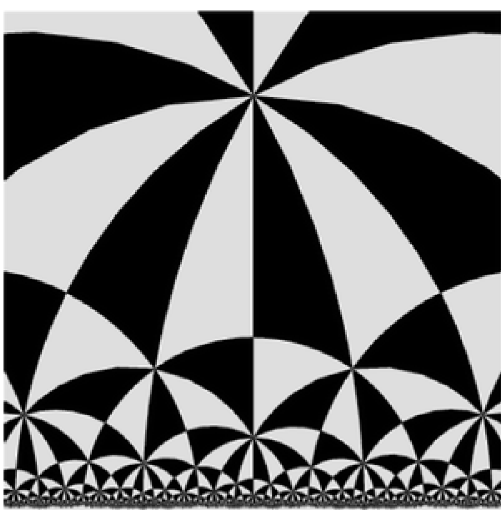
Isometries in the Euclidean plane

- **Question 5:** what are some isometries?
- **Problem 6:** You are given a triangle two of whose sides are equal (isoceles). Prove that two of the angles of the triangle are equal as well.

Explore the following by sliding around one piece of paper on top of another one:

- **Problem 7:** Say p gets mapped to itself by an isometry. What can this isometry be?
- **Problem 8:** Pick a point p in the plane and say it gets mapped to some point q by an isometry. Where does the circle of radius 1 around p get mapped to? Pick a point p' on this circle and pick a point q' for it to get mapped to. Is the behavior of the isometry on the rest of the plane fixed now?
- **Problem 9:** Any isometry of the plane is determined by what it does to 3 points.
- **Problem 10:** Prove that two distinct circles intersect in at most two points. Hint: prove the stronger statement that given 3 points there is exactly one circle containing all of them. Do you see why this implies what we want to prove?

Hyperbolic geometry

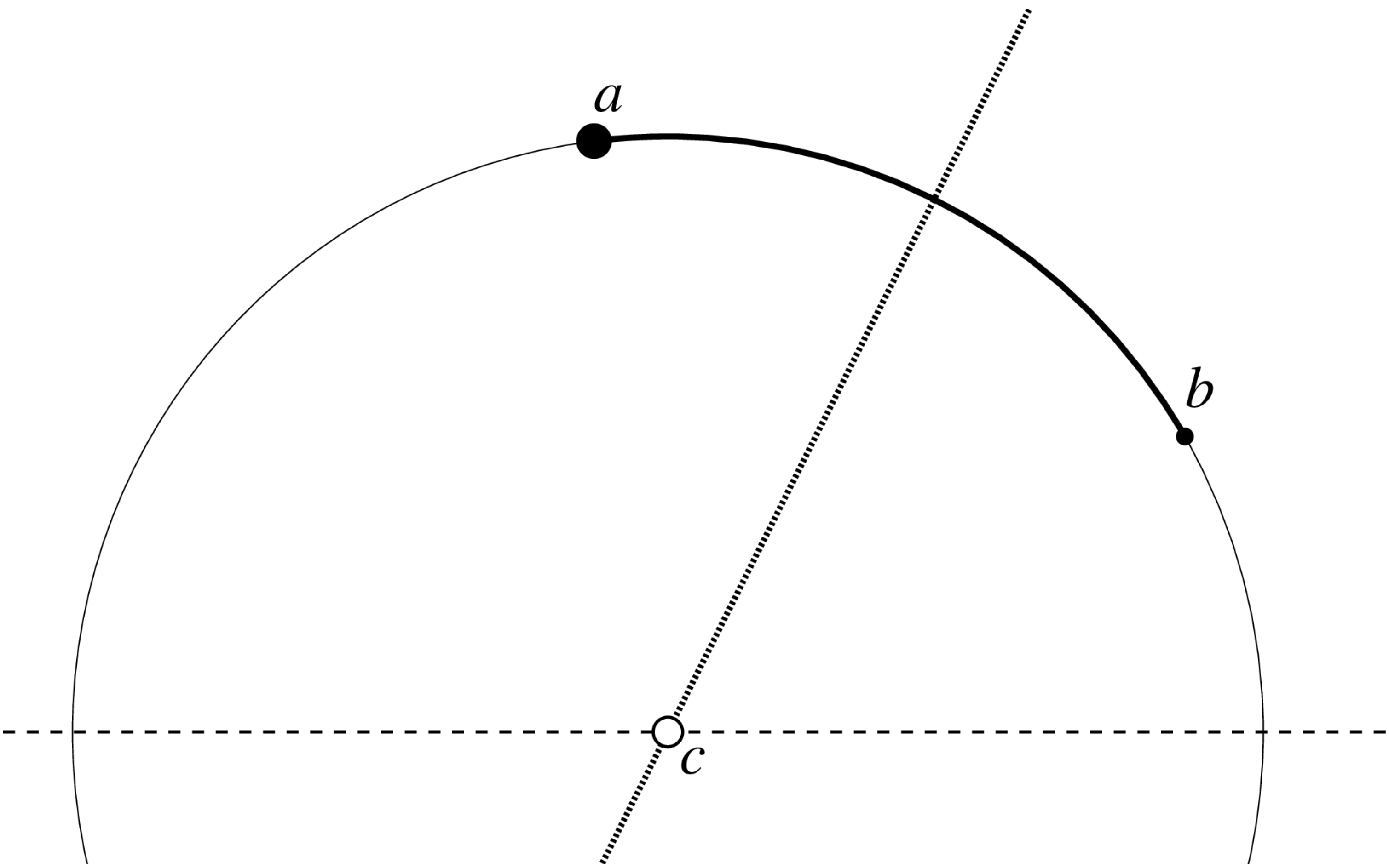


The true size of a figure at height y is $1/y$ its apparent size. Or put another way, figures at height y are drawn scaled by y. So figures at height 2 are half as big as they appear to be (because they are drawn twice as big as they should), figures at height $1/2$ are twice as big as they appear to be etc.

How do we find hyperbolic lines (besides the vertical lines) which are arcs of semi circles with center on the boundary? Here is how.

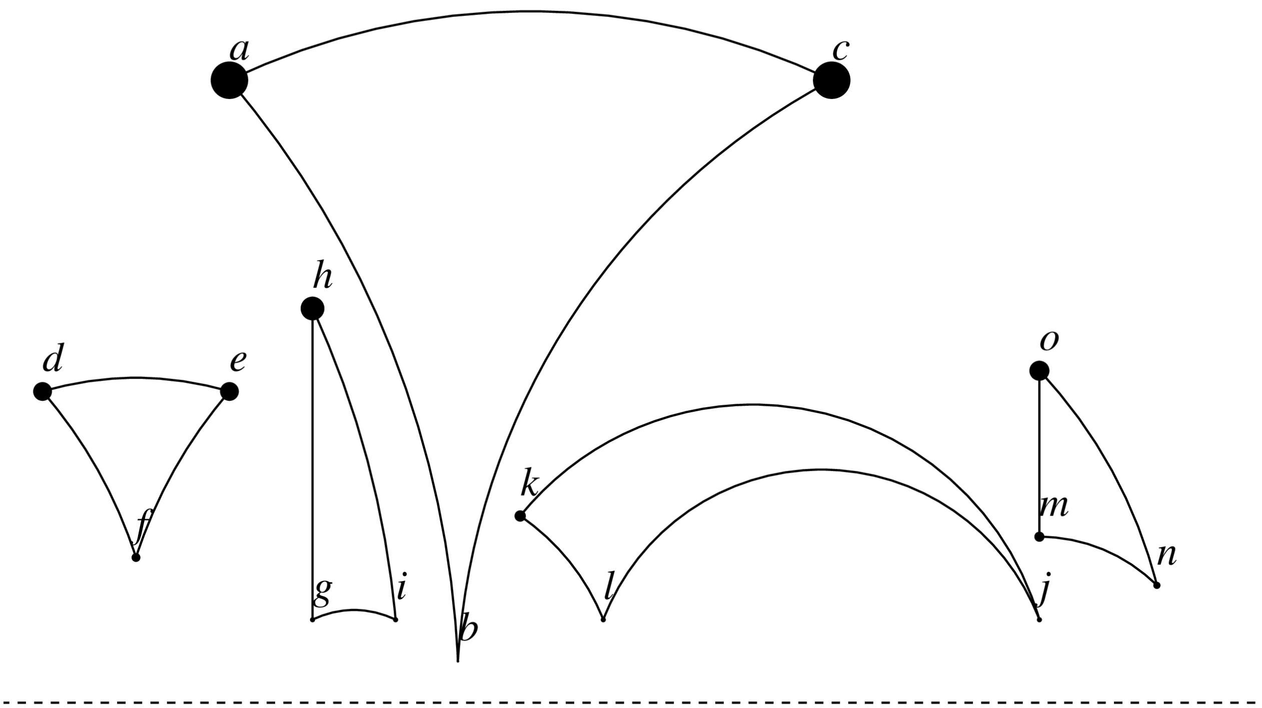
Say we are given points a and b. We start by drawing the line equidistant from a and b, the one that cuts space in half between them. This is illustrated as the line with the small dashes.

This hits the boundary (the x axis) at some point c. That point c is the the center of the circle which serves as the straight line between a and b. Remember that everything below the boundary (the horizontal dashed line) is not part of the hyperbolic plane and is only there to help us draw.

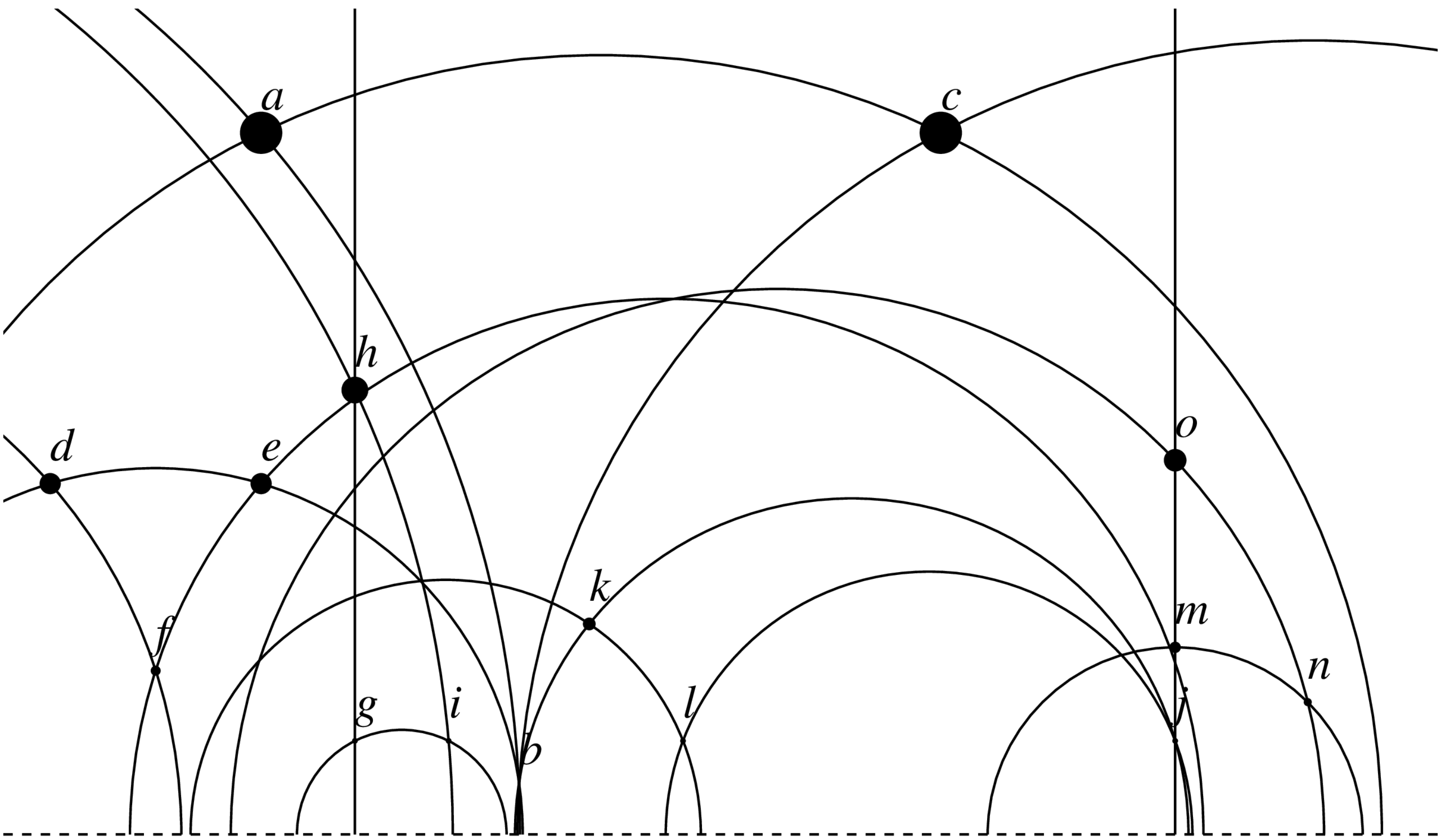


A "triangle" is 3 points connected by hyperbolic lines. Here's a few triangles.
Angles are measured by measuring the angle between the tangents to the circles.

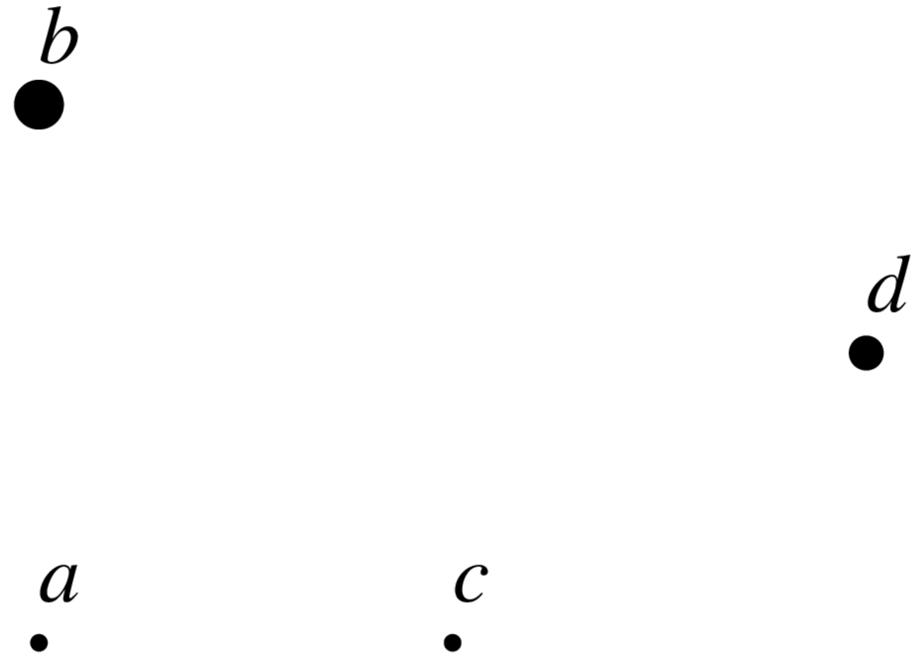
Question: what do you think the angles in each of these triangles add up to?



Here's the same points with all the line segments extended to full lines so you get a sense of what lines look like. You can also see some hyperbolic quadrilaterals like the one containing a and c.

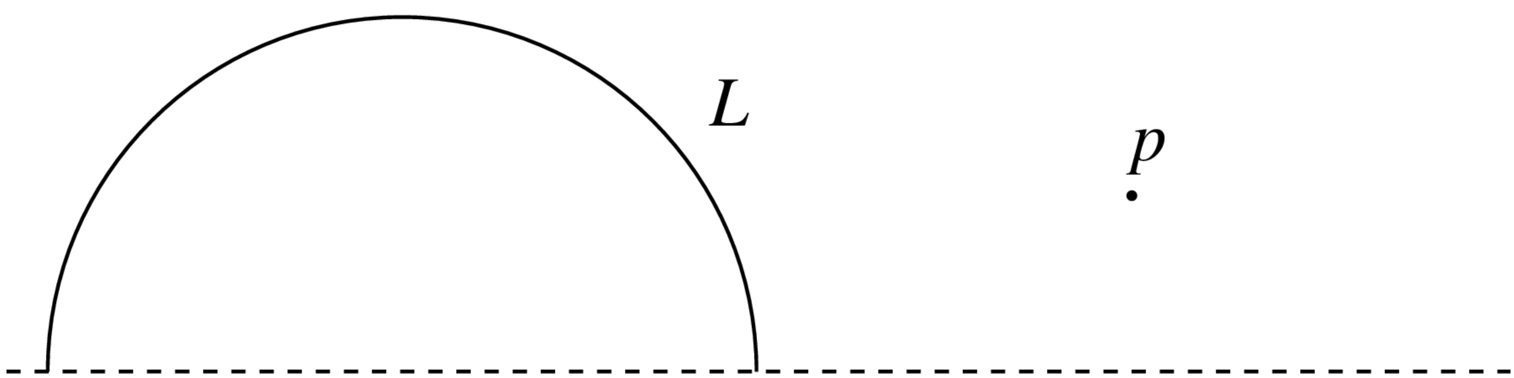


Activity 11: draw hyperbolic lines between some points. You can find the center of the semicircle needed by following the steps in the above diagram: draw the line that cuts space in half between the two points, and find where it intersects the boundary line.



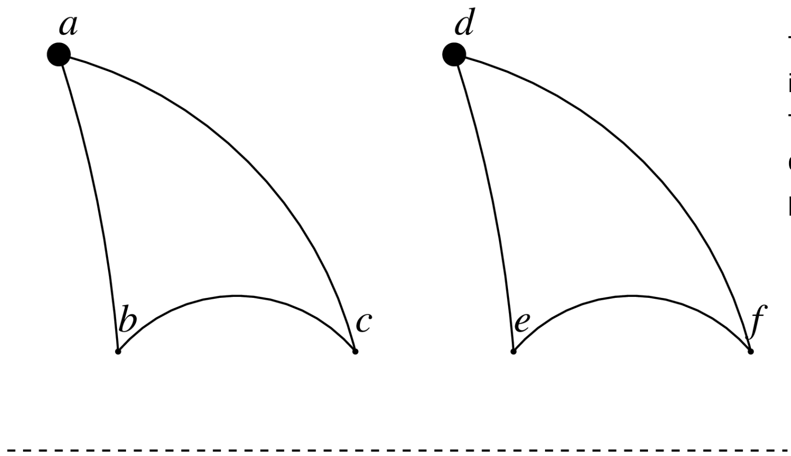
In Euclidean geometry, we saw that for any line L and point p there was exactly one line through p parallel to L. Parallel here should be understood as meaning disjoint. In hyperbolic geometry this is not the case.

Problem 12: given a hyperbolic line L and a point p, find more than one line through p that doesn't intersect L.



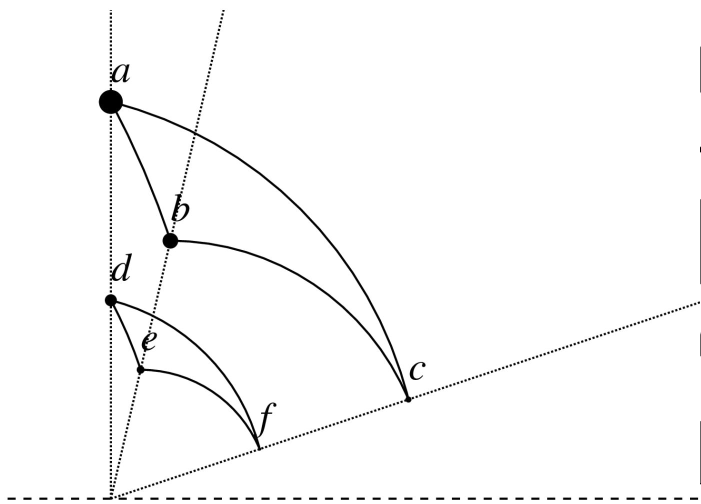
Isometries

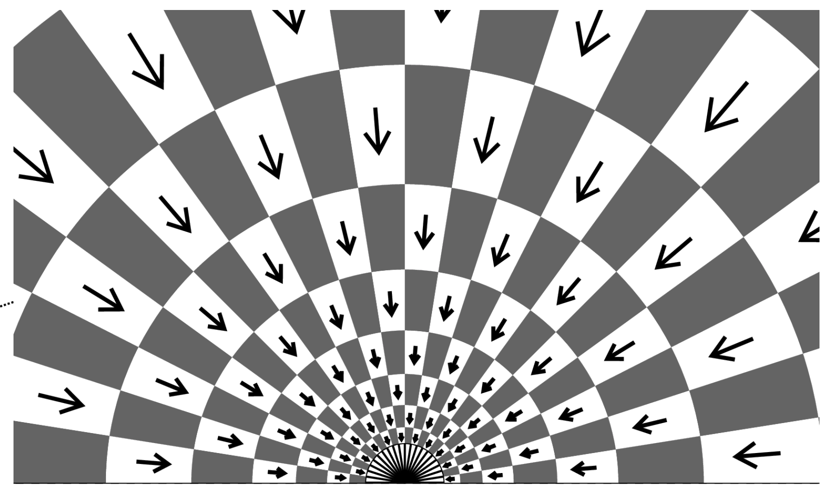
In the hyperbolic plane, these triangles have the same size



There is an isometry taking one to the other. The isometry is just horizontal shifting (translation). This is an isometry because the distortion only depends on the height, so shifting left and right preserves size.

These two triangles on the left also have the same size. Can you see why? The dashed lines give a clue.

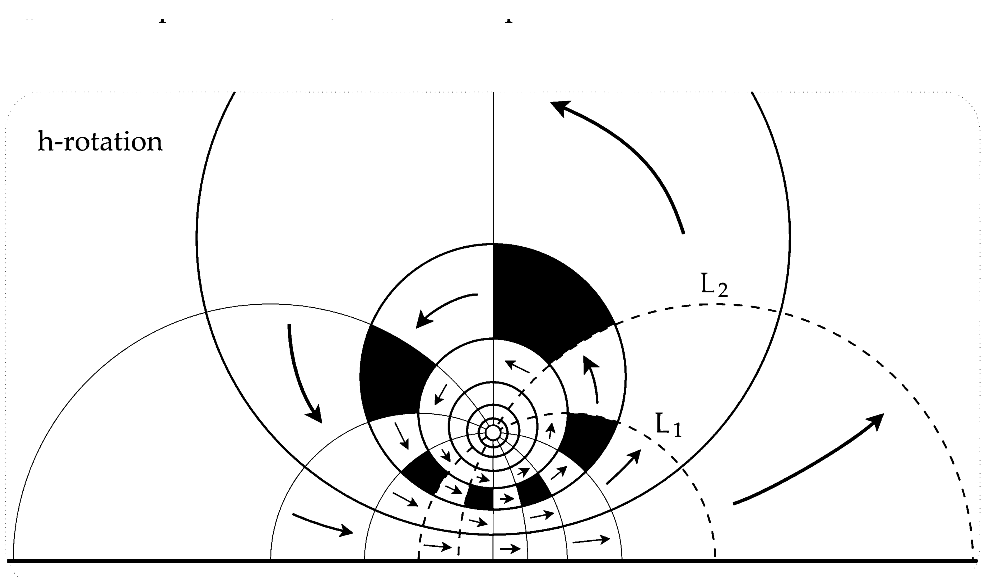




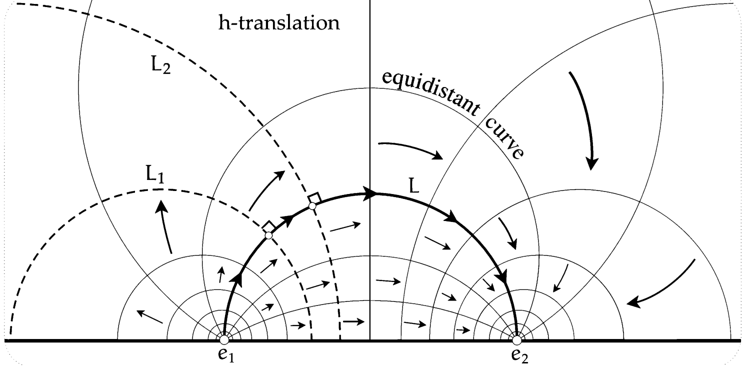
On the right is an illustration of the sort of isometry taking one of these triangles to the other. It is a dilation, a scaling of the whole plane centered at a point on the boundary.

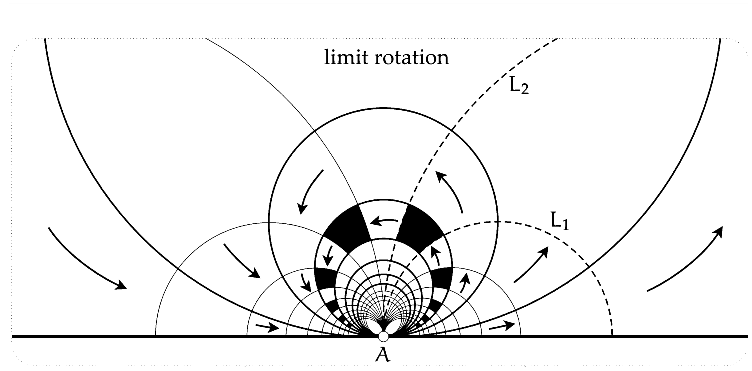
Problem 13: Explain why dilations being isometries is equivalent to saying that at height y distances are $1/y$ times what they appear to be.

There are also rotations in the hyperbolic plane. Here is an illustration of a rotation around a point.



We also have the analogue of translations





And funky things called limit rotations, or parabolic isometries. Actually, we already saw one of these: the horizontal shifts are an instance. Can you see why?

Hyperbolic geometry, just like Euclidean geometry, has reflections too. Reflections work by circle inversion.

Hyperbolic isometries can be thought about in terms of the number of *fixed points* they have on the boundary. **How many fixed points do these each of these isometries (translation, rotation, limit rotation) have on the boundary?**

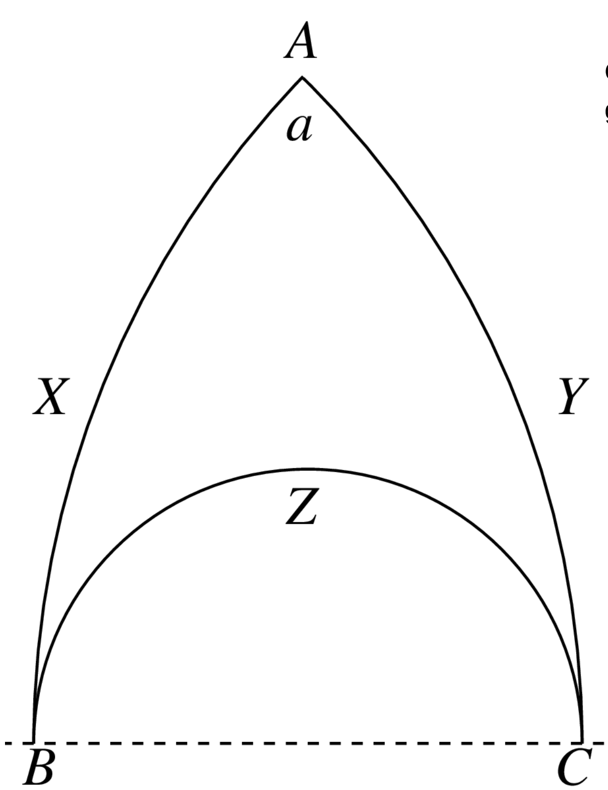
More problems

Activity 14: Draw some hyperbolic triangles and then use a dilation to draw one of the same size (even though it may look smaller or larger).

Problem 15: In the hyperbolic plane, you can make a triangle with any degree measurement as long as it adds up to less than 180. Prove this means you can make a hexagon with all angles equal to 90.

Challenge 16: given any 3 angles a, b, c, construct a triangle with the given angles.

Here is a hint on how to do it. First, pick a point A and draw lines X and Y coming out of A with angle a between them. These lines hit the boundary at points B and C. Now draw the line Z going from B to C. What angle does Z make with X and with Y. Imagine what happens as you move B up X and move C up Y.



Other resources: geometrygames.org, www.geogebra.org/m/DBXvbruR, "The Shape of Space" by Jeff Weeks (book)