Complex numbers.

Definition. A *complex number* has the form $x + \mathrm{i}y$ where x, $y \in \mathbb{R}$ and i is the *imaginary unit*, which satisfies $\mathrm{i}^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$. The *real part* $\mathrm{Re}(z)$ of $z := x + \mathrm{i}y$ is x and its *imaginary part* $\mathrm{Im}(z)$ is y. The *complex conjugate* $\overline{z}$ of z is $x - \mathrm{i}y$. The *modulus* of z is $\sqrt{x^2 + y^2}$.

The sum of two complex numbers, $x_1 + \mathrm{i}y_1$ and $x_2 + \mathrm{i}y_2$ is by definition the complex number $(x_1 + x_2) + \mathrm{i}(y_1 + y_2)$ and the product of $x_1 + \mathrm{i}y_1$ and $x_2 + \mathrm{i}y_2$ is the number $(x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1)$.

Polar representation of a complex number. Any complex number z can be written in the form $\varrho e^{\mathrm{i}\phi}$ where ϱ is the modulus of z and $\phi \in [0, 2\pi)$ is its *argument.*

Euler's formula. $e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi$.

De Moivre's formula. $(\cos\phi + \mathrm{i}\sin\phi)^n = \cos n\phi + \mathrm{i}\sin n\phi$.

Fundamental theorem of algebra. If $n \geq 1$ and a_0, a_1, …, a_n are complex numbers with $a_n \neq 0$, then there is a complex number z that satisfies the equation

$$a_0 + a_1 z + \cdots + a_n z^n = 0.$$

(In fancy terms, the field $\mathbb{C}$ is algebraically closed.)

Examples.

1. Find all $z \in \mathbb{C}$ that satisfy $\mathrm{Re}(z) = \mathrm{Im}(z)$ and
$$|z| + |z + 14| = 28.$$
2. Find all fifth roots of unity.
3. Find all solutions of the equation $(z - 1)^n = z^n$.
4. Prove the trigonometric identity
$$\cos^n\theta = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)\theta).$$
Hint: recall the *Binomial theorem:*
$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$
5. Prove the *Multisection formula:* if $f(x) = \sum_k a_k x^k$, then
$$\sum_{k \equiv r \pmod m} a_k x^k = \frac{1}{m}\sum_{s=0}^{m-1}\varepsilon^{-rs}f(\varepsilon^s x) \qquad \text{where} \quad \varepsilon \text{ is a primitive } m\text{th root of 1.}$$
The latter means that $\varepsilon^m = 1$ but $\varepsilon^j \neq 1$ for $j = 1, \ldots, m-1$.

6. Evaluate

$$\sum_{k\equiv 1 \pmod 3} \binom{n}{k} \qquad \text{and} \qquad \sum_{k\equiv 2 \pmod 3} \binom{n}{k}.$$

7. Prove that the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is *not* divisible by 5 for any integer $n \geq 0$.