BMC **Number Theory meets Counting** **December 2024**

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Here are a collection of interesting questions that lie at the intersection of number theory and combinatorics (counting).

1 It is easy to see that the product of two consecutive integers is even (in other words, is divisible 2). What can you say about the product of n consecutive integers?

The number of divisors and the d-function

2 Let $d(n)$ denote the number of (positive) divisors of n, including 1 and n. Thus $d(1) = 1$, and $d(p) = 2$ for any prime p.

(a) If p is prime, find $d(p^m)$.

(b) What is $d(2024)$? What is $d(2025)$?

(c) Prove that $d(n)$ is odd if and only if n is a perfect square.

(d) Find the smallest n that satisfies $d(n) = 10$.

Coprime numbers and the Euler ϕ-function

3 Two integers a, b are called *relatively prime* or *coprime* if their greatest common divisor is 1. Equivalently, a and b are coprime if they have no common prime factors, and we use the notation $a \perp b$.

(a) Let $\phi(n)$ denote the number of positive integers less than or equal to n which are coprime to n. Verify that $\phi(p) = p - 1$ if p is prime, and $\phi(10) = \phi(12) = 4$.

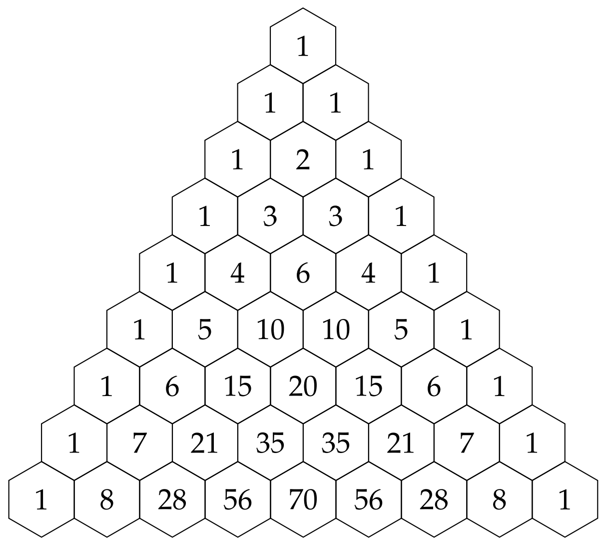
(b) Find $\phi(1001)$.

(c) How many numbers under a million are coprime to 1001? How many are coprime to 2002? How many are coprime to 7007?

4 Suppose we play the following game: A bucket is filled with N ping pong balls, labeled $1, 2, \ldots, N$, where $N = 10^{100}$. The bucket is shaken, you pick a ball, record its number as a, then you return the ball and repeat the process, getting b. If $a \perp b$, you win \$100. You can play this game as often as you want, but each time you play, you have to pay an admission fee. Suppose that several of you are competing in an auction, and the winner of that auction gets to play the game as often as they want, but each time has to pay the winning auction amount. Clearly, if you won the auction with a low amount like \$10, you would be in great shape, and it is obvious that you would not want to win the auction with an amount that is close to \$100 (or higher). What is the *correct* price to aim for?

Pascal's triangle

5 Recall *Pascal's triangle*, where we use the convention of labeling rows and columns starting from zero. For example, the number in row 7, column 2 is 21, and we write this $\binom{7}{2} = 21$.



(a) Suppose you looked at the first trillion or so rows of Pascal's triangle and you randomly picked an element. Is it more likely to be odd or even?

(b) Given arbitrary values of n, k, can we determine if $\binom{n}{k}$ is even or odd?

(c) What if, instead of odd/even, we asked about divisibility by a different prime? For example, can you tell if $\binom{2024}{1999}$ is divisible by 3 or not?

Fermat's little theorem

6 *Fermat's little theorem* says that if p is prime, and $a \not\equiv 0 \pmod{p}$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

An equivalent formulation is that

$$a^p \equiv a \pmod{p}$$

for any a. We will show that this can be proven by asking the following counting question:

> *How many different necklaces can be made using p beads, where the beads can be any of a colors?*