Berkeley Math Circle   Sept. 18, 2024   Generating Functions

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Classic problems

1 *Making Change*. How many different ways can you change a dollar with coins?

2 *Funny Dice*. A standard die is labeled $1,2,3,4,5,6$ (one integer per face). When you roll two standard dice, it is easy to compute the probability of the various sums. For example, the probability of rolling two dice and getting a sum of 2 is just $1/36$, while the probability of getting a 7 is $1/6$.

Is it possible to construct a pair of "nonstandard" dice (possibly different from one another) with positive integer labels that nevertheless are indistinguishable from a pair of standard dice, if the sum of the dice is all that matters? For example, one of these nonstandard dice may have the label 8 on one of its faces, and two 3's. But the probability of rolling the two and getting a sum of 2 is still $1/36$, and the probability of getting a sum of 7 is still $1/6$.

3 *Weights and Balances*. Let n be any positive integer. Show that the set of weights

$$1,3,3^2,3^3,3^4,\ldots$$

grams can be used to weigh an n-gram weight (using both pans of a scale), and that this can be done in exactly *one* way.

4 *Arithmetic Progressions*. A *partition* of a set is a decomposition of the set into mutually disjoint subsets. (There is another mathematical meaning of partition; see problem 11 on p. 4.) For example, the positive integers can be partitioned into evens and odds. Note also that both the evens and the odds are infinite arithmetic progressions. This leads to some interesting questions.

(a) Are there other ways to partition the positive integers into two or more arithmetic progressions?

(b) Can you do it so that the arithmetic progressions have *distinct* common differences?

Important formulas, mostly binomial

5 *The Fundamental Formula*. Recall (better yet, prove!) that

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots.$$

6 *Fun with the Binomial Theorem*. Recall (or better yet, prove!) the Binomial Theorem:

$$(1+x)^n = 1+nx+\binom{n}{2}x^2+\cdots+\binom{n}{k}x^k+\cdots+x^n,$$

for positive integer values of n.

(a) Plug $x = 1$ into the binomial theorem to get a nice formula for 2^n. Can you interpret it combinatorially?

(b) Plug $x = -1$ into the binomial theorem to get a nice formula that equals zero. Again, can you interpret it combinatorially?

(c) By considering the product $(1+x)^n(1+x)^n$, prove the beautiful formula

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

(d) In a similar way, prove the *Vandermonde Convolution Formula* (discovered several centuries before Vandermonde by Zhu Shijie in China), which states that for any positive integers $k < m, n$,

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}.$$

(e) Combining (a) and (b) above, find a formula for the sum of "every other" binomial coefficient. For example, can you find a simple formula for

$$\binom{100}{0} + \binom{100}{2} + \binom{100}{4} + \cdots + \binom{100}{100}?$$

(f) Can you generalize your method to find the sum of "every third" binomial coefficient? For example, can you compute (with a fairly simple formula)

$$\binom{100}{0} + \binom{100}{3} + \binom{100}{6} + \cdots + \binom{100}{99}?$$

Hint: complex numbers!

7 *Binomial Theorem for Negative Powers*. Recall that the Binomial Theorem says

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + x^n,$$

for positive values of n. Is there a formula for negative values of n as well? The answer, of course, is, "Yes!" Let us try to find a formula for $(1+x)^{-3}$, as an example. Follow these steps.

(a) Inspired by the Fundamental Formula, let us instead look at $(1-y)^{-3}$. We can always substitute $x = -y$ later. Now you can write

$$(1-y)^{-3} = \frac{1}{(1-y)^3} = (1+y+y^2+y^3+\cdots)^3$$

Now, suppose we wanted, say, to find the coefficient of y^7 in the above product. Write the product as

$$(1+y+y^2+y^3+\cdots) \times (1+y+y^2+y^3+\cdots) \times (1+y+y^2+y^3+\cdots),$$

where we use the $\times$ symbol to emphasize multiplication. One way to get y^7 in this product would be to take y^2 in the first sum, y^3 in the second sum, and y^2 in the third sum. In other words, the product

$$yy \times yyy \times yy,$$

where the $\times$ symbols help to remind us which term came from which sum. Of course there are other ways to get y^7. Another way is to multiply the y^5 from the first sum and the y^2 from the second sum (and the 1 from the third)

$$yyyyy \times yy \times,$$

where we leave out the 1, but include the final $\times$, since it helps us to remember which terms came from where.

Verify that *all* possible sequences using 7 ys and 2 $\times$s corresponds to a way to get a y^7 term in the product. For example, what would

$$\times \times yyyyyyy$$

mean?

(b) Recall (or prove!) that the number of ways of permuting a sequence of a 0s and b 1s is

$$\frac{(a+b)!}{a!b!} = \binom{a+b}{a} = \binom{a+b}{b}.$$

Use this to generalize the argument in (a) to show that

$$\frac{1}{(1-y)^3} = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots,$$

where

$$c_r = \binom{r+2}{r} = \binom{r+2}{2}.$$

(c) Generalize to show that, for m a positive integer,

$$\frac{1}{(1-y)^m} = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots,$$

where

$$c_r = \binom{r+m-1}{r} = \binom{r+m-1}{m-1}.$$

(d) Now plug $x = -y$ into the above formula, to get a series for $1/(1+x)^m$. Compare it with the original binomial theorem (for positive powers). What do you discover?

More Problems to Try

8 (a) How many ordered triples (x,y,z) of non-negative integers satisfy $x+y+z=15$?

(b) How many ordered triples (x,y,z) of integers greater than 3 satisfy $x+y+z=15$?

(c) How many ordered triples (x,y,z) of integers less than 9 satisfy $x+y+z=15$?

9 Show that every positive integer has a *unique* binary (base-2) representation. For example, 6 is represented by 110 in binary, since $1\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 = 6$. (This uniqueness can be proven in several ways; you are urged to try generating functions here, of course.)

10 *Generating Function Jeopardy*. For the generating functions below, find a question that the function "answers." The first problem is solved as an example.

(a) $(x^2+x^3+x^4)^7$ is the answer to the question: For each n, how many ways can we select n toys chosen from 7 types of toys, where we must pick between 2 and 4 toys of a given type?

(b) $(1-x^2)^{-3}$.

(c) $(1-x^2)^{-1}(1-x^5)^{-1}$.

(d) $(1+x^1+x^4+x^9+x^{16}+x^{25}+\cdots)^4$.

(e) $(1-x)^{-3}+(1+x)^{-3}$.

(f) $1/(1-f(x))$, where $f(x)=(x+x^2+x^3+x^4+x^5+x^6)$.

(g) $f(f(x))$, where $f(x)$ is defined in the previous problem.

11 *Partitions*. Given a positive integer n, a **partition** of n is a representation of n as a sum of positive integers. The order of the summands does not matter, so they are conventionally placed in increasing order. For example, $1+1+3$ and $1+1+1+1+4$ are two different partitions of 5.

(a) Show that for each positive integer n, the number of partitions of n into unequal parts is equal to the number of partitions of n into odd parts. For example, if $n=6$, there are 4 partitions into unequal parts, namely

$$1+5, \quad 1+2+3, \quad 2+4, \quad 6.$$

And there are also 4 partitions into odd parts,

$$1+1+1+1+1+1, \quad 1+1+1+3, \quad 1+5, \quad 3+3.$$

(b) Show that the number of partitions of a positive integer n into parts that are not multiples of three is equal to the number of partitions of n in which there are at most two repeats. For example, if $n=6$, then there are 7 partitions of the first kind, namely

$$1+1+1+1+1+1, \quad 1+1+1+1+2, \quad 1+1+2+2,$$
$$1+1+4, \quad 1+5, \quad 2+2+2, \quad 2+4;$$

and there are also 7 partitions of the second kind,

$$1+1+4, \quad 1+1+2+2, \quad 1+2+3, \quad 1+5, \quad 2+4, \quad 3+3, \quad 6.$$

Can you generalize this problem?

12 Alberto places N checkers in a circle. Some, perhaps all, are black; the others are white. (The distribution of colors is random.) Betül places new checkers between the pairs of adjacent checkers in Alberto's ring: she places a white checker between every two that are the same color and a black checker between every pair of opposite color. She then removes Alberto's original checkers to leave a new ring of N checkers in a circle.

Alberto then performs the same operation on Betül's ring of checkers following the same rules. The two players alternately perform this maneuver over and over again.

Show that if N is a power of two, then all the checkers will eventually be white, no matter the arrangement of colors Alberto initially puts down. Are there any interesting observations to be made if N is not a power of two?