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BMC Intermediate I
Kaprekar Constant
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Problem: Let $K = \{1, 2, ..., 2n\}$. Split K into two disjoint sets A and B of equal size n. So, $AUB = K$. If $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ where $a_i > a_j$ and $b_i < b_j$ for $1 \leq i < j \leq n$, what is $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n|$?

D R Kaprekar (1905 to 1986) was an Indian mathematician known for his significant contributions to recreational mathematics. Kaprekar made remarkable discoveries that earned him a place in mathematical history.

1. Kaprekar Constant (6174): Kaprekar is best known for the discovery of Kaprekar's constant, 6174. This number arises from an iterative process that takes any four-digit number, orders its digits in descending and ascending order, subtracts the smaller from the larger, and repeats the process.

E.g.: If we start with 9235,
$9532 - 2359 = 7173$
$7731 - 1377 = 6354$
$6543 - 3456 = 3087$
$8730 - 0378 = 8352$
$8532 - 2358 = 6174$
$7641 - 1467 = 6174$

And the process will repeat the same calculation again and again. We have reach the Kaprekar Constant.

The following diagram proves the following when the iterative process is applied to any 4 digit number with at least two different digits:

1) It needs at most 7 steps to reach the Kaprekar Constant

2) There is no circularity (i.e. loop of length ¿ 1) and the Kaprekar constant is reached with every starting number

3) The Kaprekar Constant is unique

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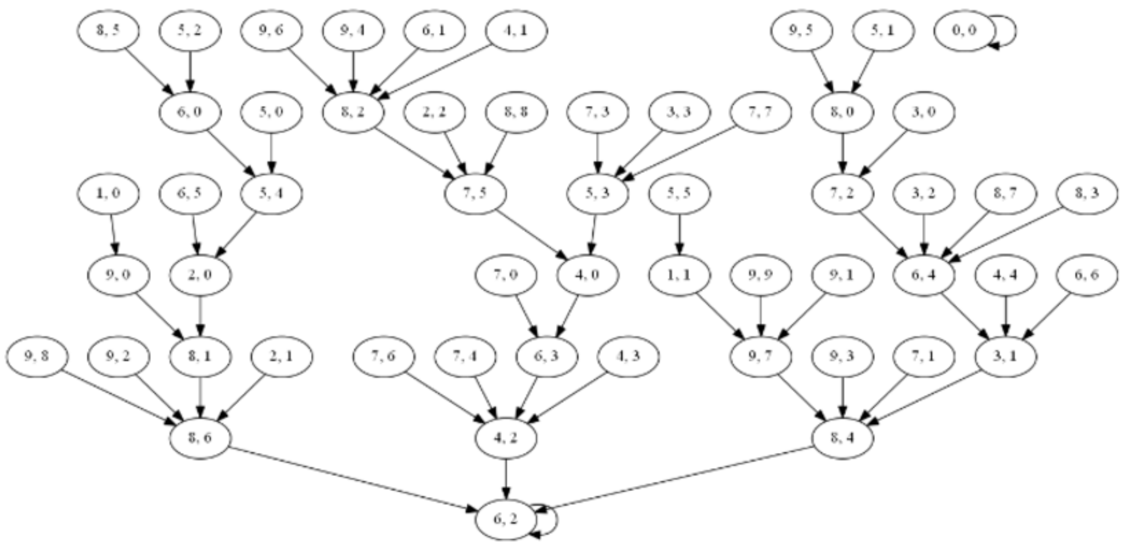
Suppose that our digits are $a \geq b \geq c \geq d$: then Kaprekar's routine generates $999(a-d)+90(b-c)$ as the next step. Since the inequalities tell us that $b-c \leq a-d$ we can easily demonstrate that there are at most 55 equivalence classes, being the 10th triangle number. That's a moderate improvement over 70.

A bit of calculation gives the graph:



Share Cite Follow   edited Sep 16, 2013 at 17:20   answered Sep 16, 2013 at 14:24
Peter Taylor
13.5k 1 31 51

Figure 1: Enter Caption

The above proof is from StackExchange answered by Peter Taylor.

We will show the uniqueness by considering cases with a little analysis.

We will only consider the numbers with at least two different digits since other numbers produce 0, which is not interesting.
Claim: 1) If the digits of Kaprekar Constant are $a \geq b \geq c \geq d$, then $b \neq c$.
2) For a number with digits $a \geq b \geq c \geq d$, if $b = c$, the Kaprekar step produces a number with 27 as the sum of the digits with 9 and 9 as middle digits, while if $b \neq c$, it produces a number with 18 as the sum of the digits.
3) A number with $a \geq b > c \geq d$ that is obtained by Kaprekar step has $a + d = 10$ and $b + c = 8$.
4) Using these properties, we need to consider 21 cases that are quick to work on.
Look up on-line articles if you are interested to find out what happens if the same process is applied to 3-digit or 5-digit numbers.

Other contributions to Mathematics:

2. Kaprekar Numbers: Kaprekar numbers are another of his discoveries. A Kaprekar number is a number n for which the square of n when split into two parts, adds up to n itself.
For example: 45 x 45 = 2025, 20 + 25 = 45. Numbers such as 9, 45, 297, and 703 are examples of Kaprekar numbers.

3. Self Numbers (Devlali Numbers): Kaprekar introduced the concept of self numbers (also known as Devlali numbers, after the town where he lived). A self number is a number that cannot be generated by adding the digits of a smaller number to that number itself. For example, 20 is not a self number because it can be generated by 15 + (1 + 5) = 20. However, 21 is a self number, as no smaller number produces it in this way.

4. Harshad Numbers (Niven Numbers): Kaprekar also worked on Harshad numbers (sometimes called Niven numbers). These are numbers that are divisible by the sum of their digits. For instance, 18 is a Harshad number because 1 + 8 = 9, and 18 is divisible by 9.

5. Demlo numbers: Look them up!

Kaprekar's discoveries have had a lasting impact, especially in the field of number theory and recreational mathematics. His passion for exploring patterns and numerical phenomena continues to inspire mathematicians and math enthusiasts today.
Through his curiosity and dedication, Kaprekar uncovered several fascinating properties of numbers that are still studied and appreciated today.