Berkeley Math Circle

Impartial Games and the World of Nim

Games

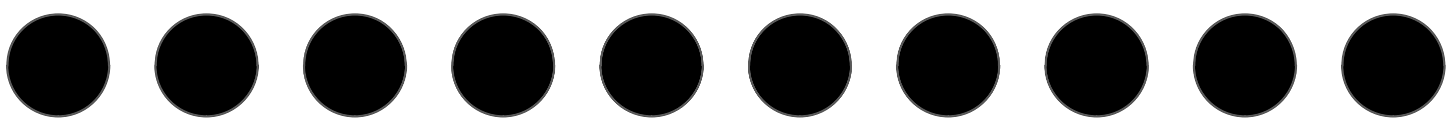
Let's play some games

- Today, we'll play several games, with different rules each time.
- Keep track of your win/loss record.
- The people with the best win/loss record will get to compete in one final game at the end of today

Base Rules

The "1-2-3" game is defined as follows:

- At the start of the game, there are 10 marbles.
- Players take turns taking either 1 marble, 2 marbles, or 3 marbles.
- The player who takes the last marble wins.



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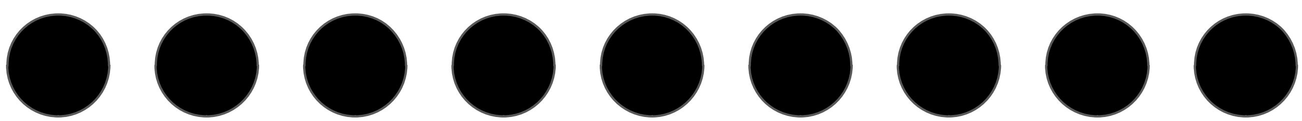

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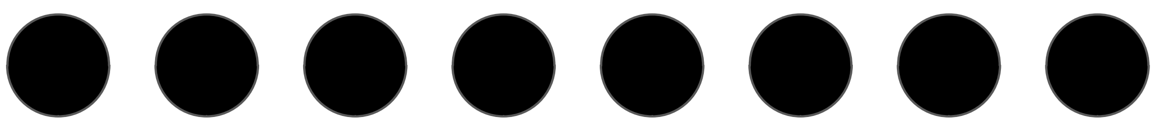

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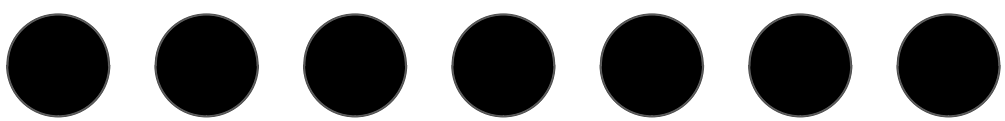

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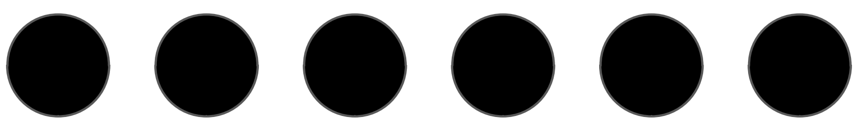

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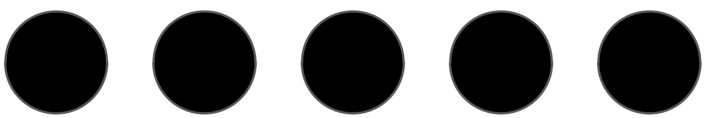

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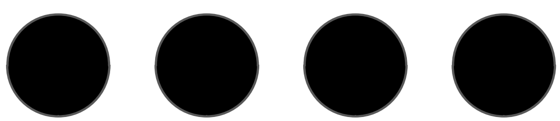

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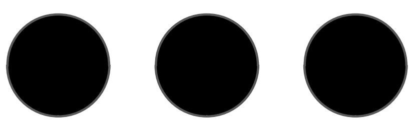

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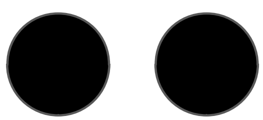

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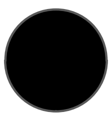

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Base Rules

The "1-2-3" game is defined as follows:

- At the start of the game, start with the number "10".
- Players take turns saying a number either 1 less, 2 less, or 3 less.
- The player who says 0 wins.
- You may not say a negative number.

Game 1: "1-2-3"

Game 1

- At the start of the game, start with the number "10".
- Players take turns saying a number either 1 less, 2 less, or 3 less.
- The player who says 0 wins.
- You may not say a negative number.

Form pairs to play at the back of the room

When you finish your game, mark the win/loss on your score sheet, then move to the front of the room

Solving a Game

Let's analyze the "1-2-3" game:

- Any game is made of several *positions.*
  - Ex. a chess position is the placement of the white and black pieces on the board.
  - In the "1-2-3" game, the position is the current number (also the number the last person said)
- "1-2-3" has no randomness, hidden information, or moves that one player can do but not the other, so if both players play perfectly, every position is either:
  - A win for the first player to move. We'll call this type of position a *winning position*
  - A win for the second player to move (and a loss for the first player). We'll call this type of position a *losing position*.
- To figure out a winning strategy for "1-2-3", let's try and figure out who wins and who loses at each position.

Solving a Game

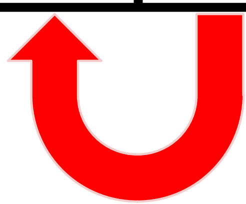
0	1	2	3	4	5	6	7	8	9	10

Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L										

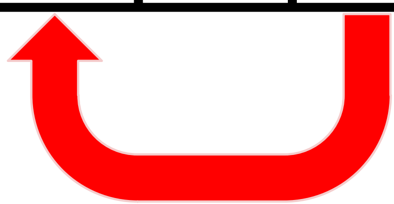
Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W									



Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W								



Solving a Game

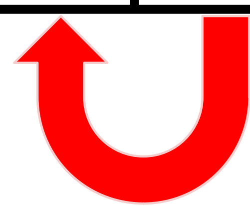
0	1	2	3	4	5	6	7	8	9	10
L	W	W	W							

Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L						

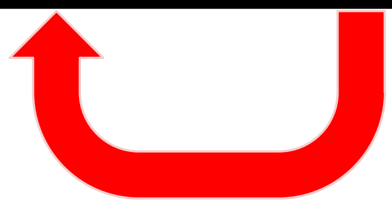
Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L	W					



Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L	W	W				



Solving a Game

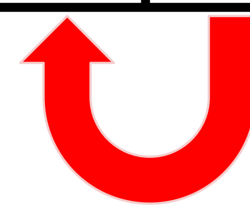
0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L	W	W	W			

Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L	W	W	W	L		

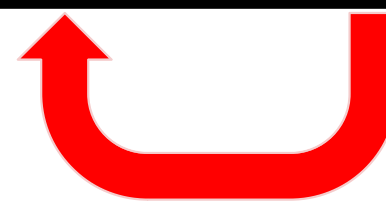
Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L	W	W	W	L	W	



Solving a Game

0	1	2	3	4	5	6	7	8	9	10
L	W	W	W	L	W	W	W	L	W	W



Solving a Game

In order to solve a game:

- Determine which positions are considered wins/losses by the game
  - If you win by moving to "0", then your opponent loses when they start their turn on "0". So "0" is considered a Loss
- For each position, look at the values of all possible moves you can make
  - If even one of them is a loss, then you are in a winning position, and can win by moving to a losing position
  - If all moves lead to winning positions, then you can't win; every move you make will cause you to lose, so you are in a losing position
- This can be used to solve ANY game, and figure out who wins in every position.

How to Decide Who Goes First

For every game from now, we'll decide who goes first in the following manner:

- I'll show the rules of the game on screen.
- Both players should try and solve the game as fast as possible
- The fastest person to say "I want to go first" or "I want to go second" gets to go first/second. The other person can choose either to start the game immediately, or keep trying to solve the game.
- The game needs to start at most one minute after the rules are shown. Otherwise both players lose.

Game 2: "1-2-3" starting with 40

Game 2

- At the start of the game, start with the number "40".
- Players take turns saying a number either 1 less, 2 less, or 3 less.
- The player who says 0 wins.
- You may not say a negative number.

When you finish your game, mark the win/loss on your score sheet, then move to the front of the room

Solving Game 2

- If at least one move goes to a loss, then your position is a win
- If all moves go to wins, then your position is a loss

For most games (like chess), there are too many positions to quickly find the value of every position.

However, sometimes, you can find a pattern of positions that contain all the losing positions.

To show that a set of positions is losing, we can show that:

- For every move made from a losing positions, there's a countermove that goes back to another losing position.
- No move goes from a losing position to another losing position

Solving Game 2

In the "1-2-3" game, the losing positions are the ones that are 0 mod 4 (they have remainder 0 when divided by 4)

- You can't go from a number that's 0 mod 4 to another number that's 0 mod 4 by subtracting 1, 2, or 3
- If you subtract 1 from a number that's 0 mod 4, you go to a position that's 3 mod 4. Counter move is to subtract 3, going back to a 0 mod 4 position
- If you subtract 2 from a number that's 0 mod 4, you go to a position that's 2 mod 4. Counter move is to subtract 2, going back to a 0 mod 4 position
- If you subtract 3 from a number that's 0 mod 4, you go to a position that's 1 mod 4. Counter move is to subtract 1, going back to a 0 mod 4 position

Game 3: "1-2-3" starting with 40, Misère

Game 3

- At the start of the game, start with the number "40".
- Players take turns saying a number either 1 less, 2 less, or 3 less.
- The player who says 0 **loses**.
- You may not say a negative number.

When you finish your game, mark the win/loss on your score sheet, then move to the front of the room

Solving Game 3

The losing positions are the ones that are 1 mod 4.

Play the same game as before, but aim to say "1" instead of "0", which forces your opponent to say "0"

Game 4: "1-3-7-15-31"

Game 4

- At the start of the game, start with the number 123,456,789.
- Players take turns saying a number either 1,3,7,15, or 31 lower.
- The player who says 0 wins.
- You may not say a negative number.

When you finish your game (or both agree on who would win), mark the win/loss on your score sheet, then move to the front of the room

Solving Game 4

All possible moves are odd numbers.

No matter how you try, the first person to move is guaranteed to win when starting from an odd number.

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Hour 2

Game 5: Prime

Game 5

- At the start of the game, start with the number 100.
- A *prime number* is a number whose only divisors are 1 and itself (except 1). The first few prime numbers are 2,3,5,7,11
- Players take turns saying a number a prime number amount lower.
- The player who says 0 **wins**.
- The player who says 1 **loses**.
- You may not say a negative number.

When you finish your game (or both agree on who would win), mark the win/loss on your score sheet, then move to the front of the room

Solving Game 5

The losing positions are the ones that are divisible by 4.

- No prime number is divisible by 4, so you can't move from a number divisible by 4 to a number divisible by 4.
- If opponent moves 2, then countermove by moving 2
- If opponent moves a prime number that's 1 mod 4, either move to 0 if possible, or by 3
- If opponent moves a prime number that's 3 mod 4, either they lost (by moving to 1), or you can move by 5 to a number divisible by 4.

Game 6: "1-2-4-8-14"

Game 6

- At the start of the game, start with the number 124.
- Players take turns saying a number either 1,2,4,8, or 14 lower.
- The player who says 0 wins.
- You may not say a negative number.

When you finish your game (or both agree on who would win), mark the win/loss on your score sheet, then move to the front of the room

Sums of games

In the original "1-2-3" game, we had one pile of 10 marbles, and could take 1, 2, or 3 marbles at a time.

Let's play two games at once!

- Begin with two piles of marbles, one with 10 marbles, and the other with 5 marbles.
- On each turn, you may take 1, 2, or 3 marbles from either pile (but not both at the same time)
- The player who takes the last marble from the last pile wins.

Sums of games

- Begin with two piles of marbles.
- On each turn, you may take 1, 2, or 3 marbles from either pile (but not both at the same time)
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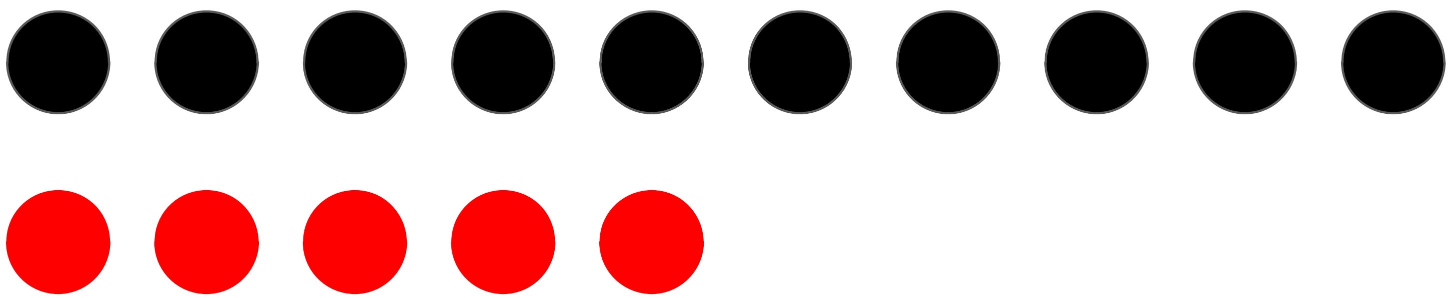

Sums of games

- Begin with two piles of marbles.
- On each turn, you may take 1, 2, or 3 marbles from either pile (but not both at the same time)
- The player who takes the last marble from the last pile wins.

Game 7: Three games of "1-2-3" at the same time

Game 7

- At the start of the game, start with three numbers 4, 38, and 41.
- Players take turns replacing **any** number with a number 1, 2, or 3 less.
- The player who replaces the last positive number with a 0 wins.
- You may not say a negative number.

When you finish your game (or both agree on who would win), mark the win/loss on your score sheet, then move to the front of the room

Analyzing Multiple Games at Once

Two key factors that apply to ALL games:

If you have two identical games, then that is a **losing** position

- Why? For every move your opponent makes, countermove by doing the same move on the other pile

If you add multiple losing games together, then that is a **losing** position

- Why? If your opponent makes a move in game 1, countermove in game 1 to get a losing position there. If your opponent makes a move in game 2, countermove in game 2 to get a losing position.

Analyzing Game 7

The game starts at 4, 38, 41

- 4 is a losing position. So if we can get (38, 41) to a losing position, then we win.
- If we take 3 from 41, we get (38, 38), which is a losing position. So *one* winning move is to take 3 from 41.
- Are there any other winning moves?
  - As it turns out, yes! We can take 1 from 38, or 1 from 4 as well. But proving that those are winning moves is a bit hard right now…