Pigeons, Holes, and the Pigeonhole Principle

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Warm up

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- I need 8 socks. With 7 or less, I could pick one of each color.

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- Pigeons $=$ socks
- Holes $=$ colors $= 7$
- To have two pigeons in the same hole (two socks of the same color), we must have $7+1=8$ pigeons (socks)

Pigeonhole Principle

If $k+1$ pigeons are put into k holes, some hole will have at least two pigeons.

- **Proof**

  Proof technique: Proof by Contradiction

  To prove the statement "if A, then B", assume B is false and show that you get a logical contradiction.

  Assume by contradiction that no hole has more than one pigeon.

  There are k holes, so there are $\leq k$ pigeons.

  But, there are $k+1$ pigeons, so this is a contradiction ($\lightning$). So, there is a hole with at least two pigeons. ☺

Example

T/F: The Pigeonhole Principle states that if the socks I have are of the 7 colors of the rainbow, if I pick 8 socks, at least two socks that I picked are yellow.

- False. At least two socks are of the same color, but they could be of any color.

T/F: The Pigeonhole Principle states that if I pick 8 socks, exactly two socks are of the same color.

- False. I could have three socks of the same color, or all socks of the same color.

Example

How many socks do I need to pick to guarantee at least 5 of the same color?

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How many socks do I need to pick to guarantee at least 5 of the same color?

- I need to pick 29 socks. If I pick 28, I might have 4 of each color.

Generalized Pigeonhole Principle (version 1)

If $(n-1)k+1$ pigeons are put into k holes, then some hole has at least n pigeons.

Example

If I pick 50 socks, how many socks of the same color am I guaranteed to have?

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- If I divide 50 socks by 7 colors, I could have 7 socks per color but would have one remaining. So, I am guaranteed at least 8 socks of the same color.
- Equivalently, $\frac{50}{7}$ rounded up makes 8.

Generalized Pigeonhole Principle (version 2)

If n pigeons are put into k holes, then some hole has at least $\lceil \frac{n}{k} \rceil$ pigeons.

- **Proof**

  A property of the ceiling function is that $\lceil x \rceil < x + 1$.

  Assume by contradiction that every hole has less than $\lceil \frac{n}{k} \rceil$ pigeons. Equivalently, the maximum number of pigeons each hole can have is $\lceil \frac{n}{k} \rceil - 1$ pigeons.

  We can find the total number of pigeons.

$$
\begin{aligned}
\#\text{ pigeons} \leq \left(\left\lceil \frac{n}{k} \right\rceil - 1 \right) k &= \left\lceil \frac{n}{k} \right\rceil \cdot k - k \\
&< \left(\frac{n}{k} + 1 \right) \cdot k - k \\
&= n + k - k \\
&= n
\end{aligned}
$$

Generalized Pigeonhole Principle

If n pigeons are put into k holes, then some hole has at least $\lceil \frac{n}{k} \rceil$ pigeons.

- **Proof**

  So, the number of pigeons is $< n$. But, we know that there are n pigeons, so $n < n$, which is a contradiction. Thus, some hole has at least $\lceil \frac{n}{k} \rceil$. ☺

Example

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- Pigeons = people = n
- Holes = # of rounds a person can play = 1 to $n-1$ = $n-1$
- By the pigeonhole principle (PHP), some hole has at least $\lceil \frac{n}{n-1} \rceil = 2$ pigeons in it. So, at least two people must play the same number of rounds. ☺

Example

Prove that an equilateral triangle cannot be completely covered by two smaller equilateral triangles.

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- Assume by contradiction that it is possible. Then, all 3 vertices of the equilateral triangle must be covered by the two smaller triangles.
- Pigeons = vertices of the larger equilateral triangle = 3
- Holes = smaller triangles = 2
- By the PHP, some hole has at least $\lceil \frac{3}{2} \rceil = 2$ pigeons in it, meaning one of the smaller triangles must contain at least 2 vertices of the larger triangle in it.
- This is a contradiction since the two vertices are a side length apart, and the farthest away two points can be in an equilateral triangle is its side length. The smaller triangles have a strictly smaller side length than the larger equilateral triangle, so two vertices cannot lie in one smaller triangle. ☺

Example

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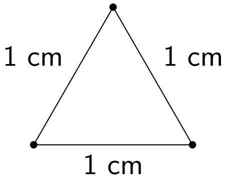
I have an arbitrarily large canvas completely covered in blue and yellow paint. Prove that there are two points of the same color on the canvas that are exactly 1 cm apart.

- Construct an equilateral triangle of side length 1cm and place it over the painting.



- Pigeons = vertices of the triangle = 3
- Holes = colors = 2
- By the PHP, at least $\lceil \frac{3}{2} \rceil = 2$ vertices must be of the same color. Any two vertices of the equilateral triangle are exactly 1 cm apart. ☺

Example

I cover the canvas in blue, yellow, and green paint. Prove that there are two points of the same color on the canvas that are exactly 1 cm apart.

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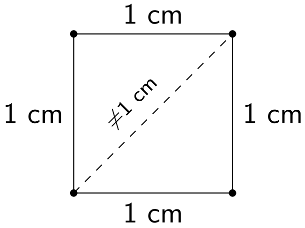
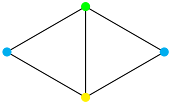
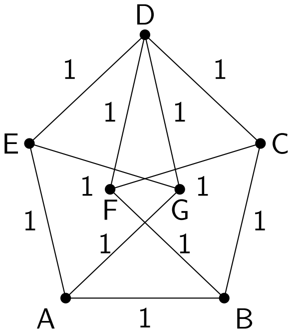
- Does a square work?



- We want to find a different shape.



Example



Moser Spindle

- Pigeons = vertices = 7
- Holes = colors = 3
- By the PHP, at least $\lceil \frac{7}{3} \rceil = 3$ vertices of the figure have the same color.
- Of any 3 vertices, at least 2 are 1 cm apart. ☺

Example

Suppose $a_1, a_2, \ldots, a_n$ are integers. Then some "consecutive sum" $a_k + a_{k+1} + a_{k+2} + \cdots + a_{k+m}$ is divisible by n.

Example

Suppose $a_1, a_2, \ldots, a_n$ are integers. Then some "consecutive sum" $a_k + a_{k+1} + a_{k+2} + \cdots + a_{k+m}$ is divisible by n.

- Write the consecutive sums:

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n
\end{aligned}
$$

  If one is divisible by n, we are done. Otherwise, dividing each by n leaves some non-zero remainder.

Example

- Notation: $a \equiv b \bmod m$ means a has remainder b when divided by m.
- Let $s_1 \equiv r_1 \bmod n$, $s_2 \equiv r_2 \bmod n$, $\ldots$, $s_n \equiv r_n \bmod n$. $r_1, \ldots, r_n$ have values in the set $\{1, \ldots, n-1\}$.
- Pigeons = possible remainders $\{r_1, \ldots, r_n\} = n$
- Holes = possible values of remainders = $\{1, \ldots, n-1\} = n-1$
- By the PHP, at least $\lceil \frac{n}{n-1} \rceil = 2$ remainders share the same value. Let these be r_i and r_j, with $j > i$.
- s_i and s_j have the same remainder when divided by n, so $s_j - s_i = a_{i+1} + a_{i+2} + \cdots + a_j$ is a consecutive sum divisible by n. ☺

Example

[Sun Tzu's Theorem] If m and n are relatively prime, and $0 \leq a < m$ and $0 \leq b < n$, then x exists such that $x \equiv a \bmod m$ and $x \equiv b \bmod n$.

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- Consider the integers $a, a+m, a+2m, \ldots, a+(n-1)m$. When divided by m, each has remainder a. We want to show that one of these has remainder b when divided by n.
- Assume by contradiction that none of the integers satisfy this.

$$\begin{aligned} a &\equiv r_0 \bmod n \\ a+m &\equiv r_1 \bmod n \\ a+2m &\equiv r_2 \bmod n \\ &\vdots \\ a+(n-1)m &\equiv r_{n-1} \bmod n \end{aligned}$$

Example

- No remainder can be b, so the possible remainders are $\{0, 1, \ldots, b-1, b+1, \ldots, n-1\}$. There are $n-1$ possible remainders, these will be the pigeons.
- There are n values of r_i, these will be the holes.
- By the PHP, at least $\lceil \frac{n}{n-1} \rceil = 2$ remainders have the same value. Let these be $r_i = r_j = r$, and WLOG, let $j > i$.

$$a + im = k_1 n + r \qquad a + jm = k_2 n + r$$

Combine the equations.

$$\begin{gathered}(a + jm) - (a + im) = (k_2 n + r) - (k_1 n + r) \\ m(j - i) = n(k_2 - k_1)\end{gathered}$$

m and n are relatively prime, so $n \mid (j - i)$. But j and i are distinct and from the set $\{0, 1, \ldots, n-1\}$, so $0 < j - i < n$ and n cannot divide $j - i$. $\lightning$, so some integer works.

Handouts

I have a bag containing 3 types of chocolate. What is the smallest number of chocolates I need to pick from the bag to ensure that I picked 3 of the same flavor?

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- Pigeons = number of chocolates
- Holes = types of chocolate = 3
- We want $n = 3$ of the same type, and $k = 3$.

$$(n-1)k + 1 = 2 \cdot 3 + 1 = 7$$

- I must pick 7 chocolates to ensure 3 of the same flavor. ☺

There are 39 million people in California. How many people can we guarantee have the same two-letter initials?

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- Pigeons = people in California = 39,000,000
- Holes = # of two-letter initials = $26 \cdot 26 = 676$
- By the PHP, at least $\lceil \frac{39000000}{676} \rceil = 57,693$ people share the same initials. ☺

Prove that if 101 numbers from 1 to 1000 are chosen, at least 11 of them must have the same ones digit.

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- Pigeons = numbers chosen = 101
- Holes = possible ones digits = $\{0, 1, \ldots, 9\} = 10$
- By the PHP, at least $\lceil \frac{101}{10} \rceil = 11$ numbers have the same ones digit. ☺

Nine students solved a total of 28 problems in a math olympiad. Each problem was solved by exactly one student. We know of one student who solved exactly one problem, two students who solved exactly two problems, and three students who solved exactly three problems. Prove that one student must have solved at least 5 problems.

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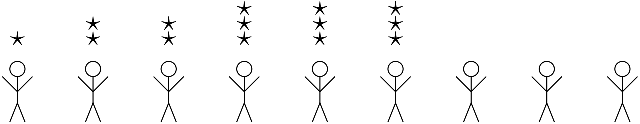

- 6 students solved $1 + 2 + 2 + 3 + 3 + 3 = 14$ questions, leaving 3 students to solve the remaining 14 problems
- Pigeons = problems = 14
- Holes = students = 3
- By the PHP, some student must have solve at least $\lceil \frac{14}{3} \rceil = 5$ problems.

Pick 7 points on the surface of a sphere. Show that at least 5 must lie on the same hemisphere of the sphere.

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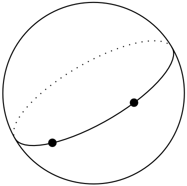
- Pick two points on the sphere, split the sphere into two hemispheres with a circle passing through those two points.



- There are 5 remaining points that must be placed.
- Pigeons = points = 5
- Holes = hemispheres = 2
- By the PHP, at least $\lceil \frac{5}{2} \rceil = 3$ points must lie on the same hemisphere. Add the two original points, this gives 5 points on the same hemisphere. ☺

Given any 9 natural numbers $(1, 2, 3, \dots)$, show that it is possible to choose 5 whose sum is divisible by 5.

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- Every natural number has a possible remainder of 0, 1, 2, 3, or 4 when divided by 5. Split the 9 numbers into sets based on what their remainders are.
- Case 1: all 5 sets have elements. Take one element from each set, the remainder is $0 + 1 + 2 + 3 + 4 = 10 \equiv 0 \mod 5$ and we are done.
- Case 2: exactly 4 sets have elements.
- Case 3: exactly 3 sets have elements.
- Case 4: exactly 2 sets have elements. Pigeons = numbers = 9 and holes = sets = 2, so by the PHP, one set has at least $\lceil \frac{9}{2} \rceil = 5$ numbers. Choose 5 numbers from the set and we are done.
- Case 5: one set has all 9 elements. Choose 5 numbers from that set, and we are done.

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- Case 4: exactly 2 sets have elements. Pigeons = numbers = 9 and holes = sets = 2, so by the PHP, one set has at least $\lceil \frac{9}{2} \rceil = 5$ numbers. Choose 5 numbers from the set and we are done.
- Case 5: one set has all 9 elements. Choose 5 numbers from that set, and we are done.

- Case 2: exactly 4 sets have elements. By the PHP, some set has at least $\lceil \frac{9}{4} \rceil = 3$ elements in it. Suppose the possible remainders are a, b, c, d, e. WLOG, suppose there are three elements in the set for a and none in the set for b. c, d, and e each have at least one element. Take $2b - a \bmod 5$, the result will be c, d, or e.
- Choose three elements from a, none from b, none from $2b - a$, and one each from the remaining two sets. The sum of these 5 elements is divisible by 5.

- Case 3: like in case 2, some set has at least $\lceil \frac{9}{3} \rceil = 3$ elements in it. WLOG, suppose there are no elements for d or e. a, b, and c can be chosen such that $d + e \equiv a + b \equiv 2c \mod 5$.
- If possible, choose 2 elements from a, 2 from b, and one from c, or 1 from a, 1 from b, and 3 from c. If either is possible, we are done.
- If both are impossible, c has 1 or 2 elements and either a or b have one element. WLOG, let it be a. Then, b has at least 6 elements. Choose 5 elements from b and we are done. ☺

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- There are $2^{10} - 2 = 1022$ distinct subsets.
- The sum of the elements in any subset must be between 10 and $90 + 91 + 92 + \cdots + 99$. There are < 1000 possible sums.
- Pigeons $=$ subsets $= 1022$
- Holes $=$ possible sums < 1000
- By the PHP, at least $\lceil \frac{1022}{1000} \rceil = 2$ subsets have the same sum. Let these subsets be A and B.
- Take the intersection of A and B, $A \cap B$, and take $A' = A - (A \cap B)$ and $B' = B - (A \cap B)$. A' and B' are disjoint and nonempty since otherwise, $A \cap B = A$ or B meaning A and B are either not distinct or have different sums.
- A' and B' are nonempty disjoint subsets of our set of numbers with equal sums. ☺

Come up with a problem that can be solved using the Pigeonhole Principle. Have a partner solve it.