Counting trees

1. A group of $n$ servers needs to be connected into a network using $n-1$ cables so that each server could communicate with every other. For example, for three servers $A, B, C$, this can be done in three different ways:

In how many ways can this be done with (a) 4; (b) 5 servers?

Recall that we defined a tree as a connected graph without cycles. Thus, Problem 1 asks to find the number of different trees with (a) 4; (b) 5 given vertices.

Last time we proved Tree Theorem stating that a tree with $n$ vertices has exactly $n-1$ edges.

2. Show that a tourist wishing to visit all 100 cities in Effiland (where, as you remember, the network of flights is a tree) can do this using no more than:

(a) 198 flights; (b) 196 flights.

There are several different ways to characterize trees.

Let $T$ be a graph with $n$ vertices (nodes). The following statements are equivalent:

(a) $T$ is a tree.

(b) For any two vertices in $T$, there is a unique path from one to the other along edges of $T$.

(c) $T$ is connected and has $n-1$ edges.

(d) $T$ has no cycles and has $n-1$ edges.

(e) $T$ is connected, but deleting any edge makes it disconnected (i.e. $T$ is minimally connected).

(f) $T$ has no cycles, but addition of any new edge creates a cycle (i.e. $T$ is maximally acyclic).

3. Verify that the statements (a)–(f) above are indeed equivalent.

4. Find all different types of trees (i.e. trees with unlabeled vertices) with (a) 6; (b) 7 vertices.

5. Find the number of trees with (a) 6; (b) 7 labeled vertices (i.e. do Problem 1 for $n = 6$, 7).