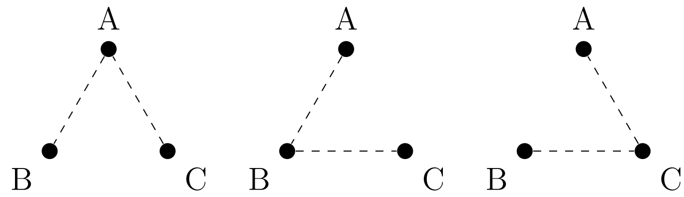
Berkeley Math Circle — January 31, 2024

Counting trees

1. A group of n servers needs to be connected into a network using $n-1$ cables so that each server could communicate with every other.

   For example, for three servers A, B, C, this can be done in three different ways:



   In how many ways can this be done with (a) 4; (b) 5 servers?

Recall that we defined ***a tree*** as *a connected graph without cycles.*

Thus, Problem 1 asks to find the number of different trees with (a) 4; (b) 5 *given* vertices.

Last time we proved ***Tree Theorem*** stating that *a tree with n vertices has exactly $n-1$ edges*.

2. Show that a tourist wishing to visit all 100 cities in Effiland (where, as you remember, the network of flights is a tree) can do this using no more than:

   (a) 198 flights; (b) 196 flights.

There are several different ways to characterize trees.

Let T be a graph with n vertices (nodes)). The following statements are equivalent:

(a) T is a tree.

(b) For any two vertices in T, there is a unique path from one to the other along edges of T.

(c) T is connected and has $n-1$ edges.

(d) T has no cycles and has $n-1$ edges.

(e) T is connected, but deleting any edge makes it disconnected (i.e. T is *minimally connected*).

(f) T has no cycles, but addition of any new edge creates a cycle (i.e. T is *maximally acyclic*).

3. Verify that the statements (a)–(f) above are indeed equivalent.

4. Find all different *types of trees* (i.e. trees with *unlabeled* vertices) with (a) 6; (b) 7 vertices.

5. Find the number of trees with (a) 6; (b) 7 *labeled* vertices (i.e. do Problem 1 for $n = 6, 7$).