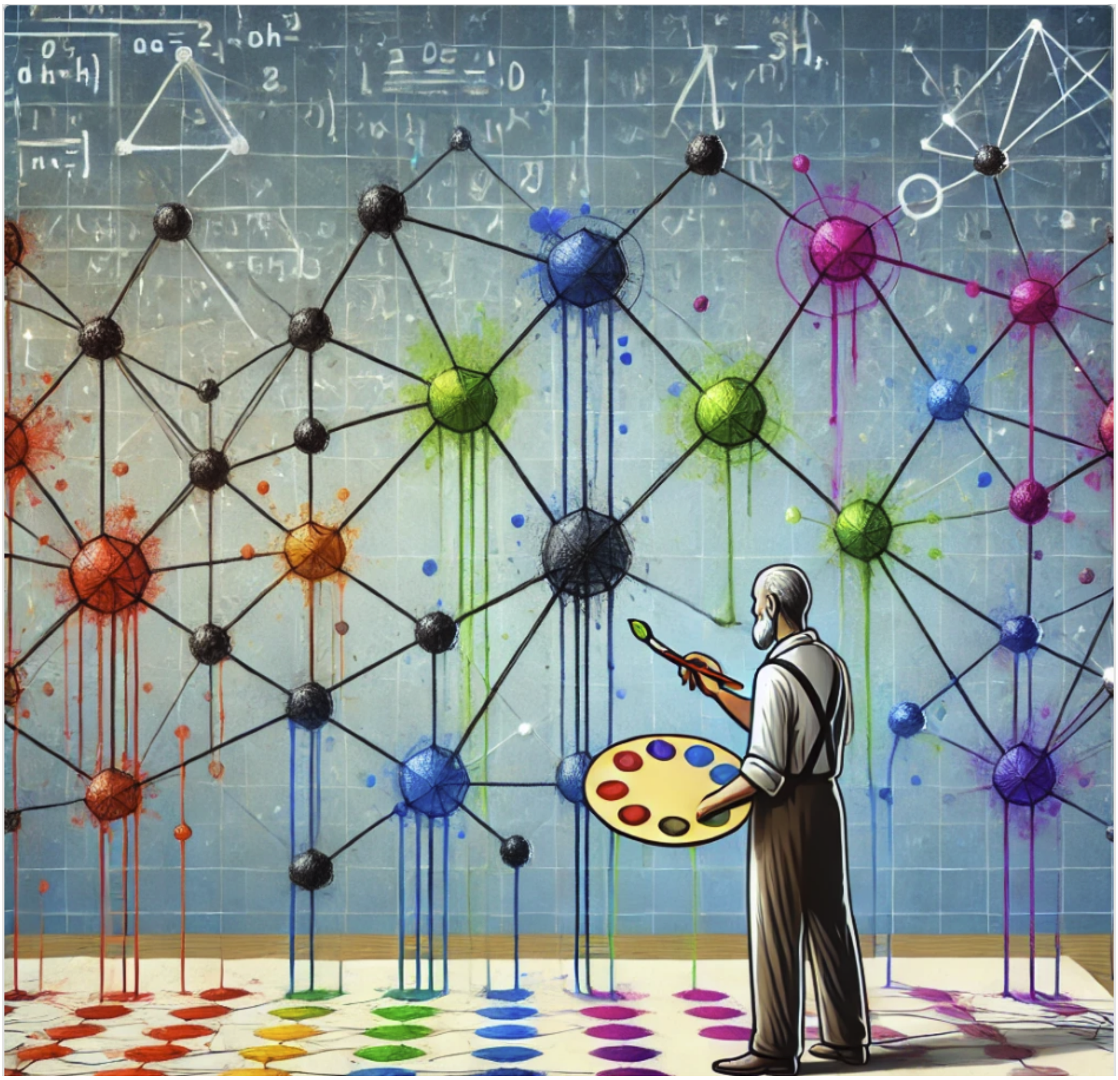
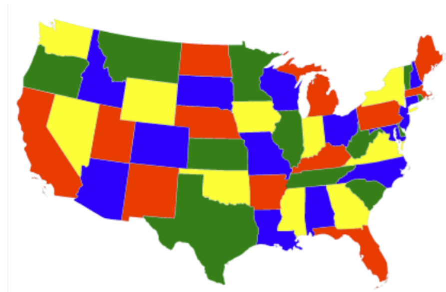
Graph Coloring



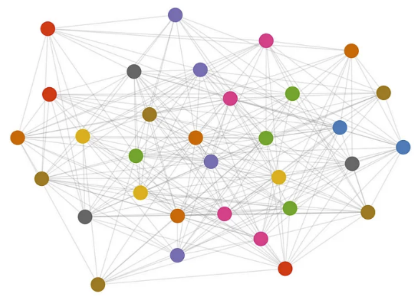
Graph theory review

1. **Terminology**: vertices, edges, degree of a vertex, paths, circuit, cycle.
2. **Types of graphs:** simple graphs, connected, disconnected, complete, bipartite, planar.
3. **Bipartite graphs.** A graph is bipartite if and only if it has no cycles of odd length.
4. **Handshake lemma.** In any graph, the sum of the degrees of all its vertices is twice the number of its edges: $d_1 + d_2 + \ldots + d_V = 2E$.
5. In any graph, the number of odd vertices (i.e. with odd degrees) is even.
6. **Euler's bridges theorem.** A connected graph has an Euler path (a path that uses each edge once) if and only if this graph has at most two odd vertices.

Map and Graph Coloring

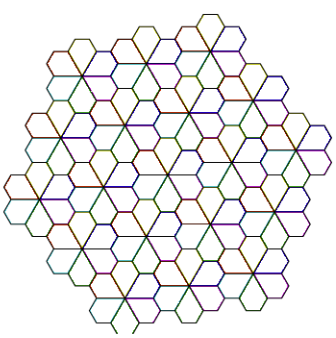
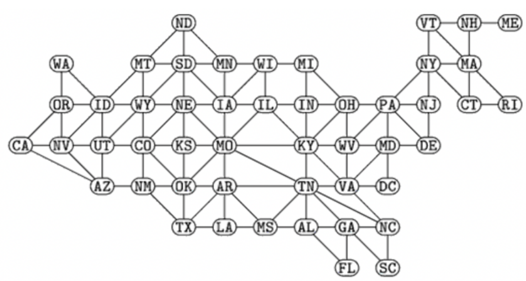


A ***coloring*** of a map is an assignment of a color to each region of the map in such a way that regions sharing parts of their boundary get different colors. The ***chromatic number*** of a map is the smallest number of colors needed to color it.



A ***coloring*** of a graph is an assignment of colors to each vertex so that no two vertices connected by an edge have the same color. The ***chromatic number*** of a graph G is the smallest number of colors needed to color it. It is denoted $\chi(G)$.

1. Find $\chi(G)$ for (a) path graph P_n; (b) cycle graph C_n; (c) complete graph K_n.
2. Describe all graphs with (a) $\chi(G) = 1$; (b) $\chi(G) = 2$.
3. Find the chromatic numbers of the following maps and graphs:

(a)  ; (b) 

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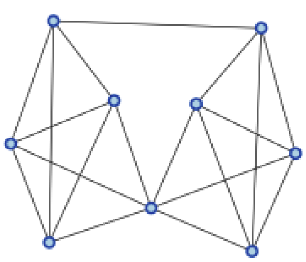
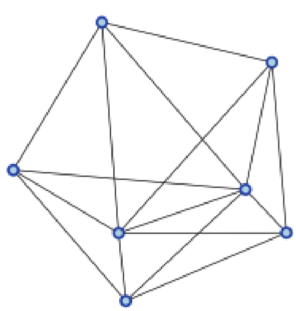
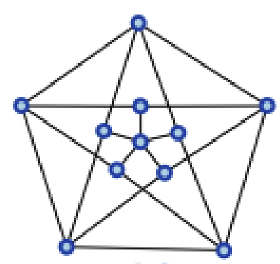
(c)  ; (d)  ; (e)  .

4. Among eight members of a math club (let's call them A, B, C, D, E, F, G, H) seven groups were formed, each to work on a certain problem in a team competition. These groups are:

$$\{A, B, C\},\ \{B, C, D\},\ \{D, E, F\},\ \{E, F, G\},\ \{A, G, H\},\ \{A, D, G\},\ \{B, F, H\}.$$

If each group has to meet during one of the time periods 1-2 pm, 2-3 pm, 3-4 pm, 4-5 pm, 5-6 pm, then what is the minimum number of time periods needed for all seven groups to meet?

5. (Bounds) Show that (a) $\chi(G) \leq \Delta(G)+1$; (b) $\chi(G) \geq \omega(G)$; (c) $\chi(G) \geq \dfrac{n}{\alpha(G)}$, where
   (a) $\Delta(G)$ is the largest degree of a vertex of G;
   (b) $\omega(G)$ is the largest number of vertices in G each connected with each other (*clique number*);
   (c) $\alpha(G)$ is the largest number of vertices with no edges between them (*independence number*).

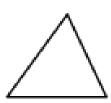
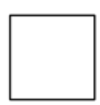
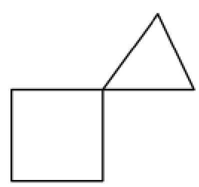
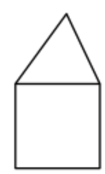
6.* (Brooks theorem) If G is connected and $\chi(G) = \Delta(G)+1$ then G is either an odd cycle or a complete graph.

The Chromatic Polynomial

The ***chromatic function*** $f_G(k)$ of a graph G is the number of ways to color G using k colors.

1. Compute $f_G(k)$ for the following graphs
   (a) the null graph N_n;
   (b) the path graph P_n;
   (c) the star graph S_n;
   (d) the triangle graph;
   (e) the complete graph K_n;
   (f) the circle graph with four vertices C_4;
   (g) the circle graph with five vertices C_5;
   (h) the circle graph C_n for all n.

2. Given two graphs G_1 and G_2, consider two new graphs: G' obtained by gluing G_1 and G_2 at one vertex and G'' be a graph obtained by gluing G_1 and G_2 at one edge. For example

   if $G_1 =$  and $G_2 =$  then $G' =$  and $G'' =$  .

   (a) Show that $f_{G'}(k) = f_{G_1}(k) \cdot f_{G_2}(k)/k$.

   (b) Express $f_{G''}(k)$ through $f_{G_1}(k)$ and $f_{G_2}(k)/k$.

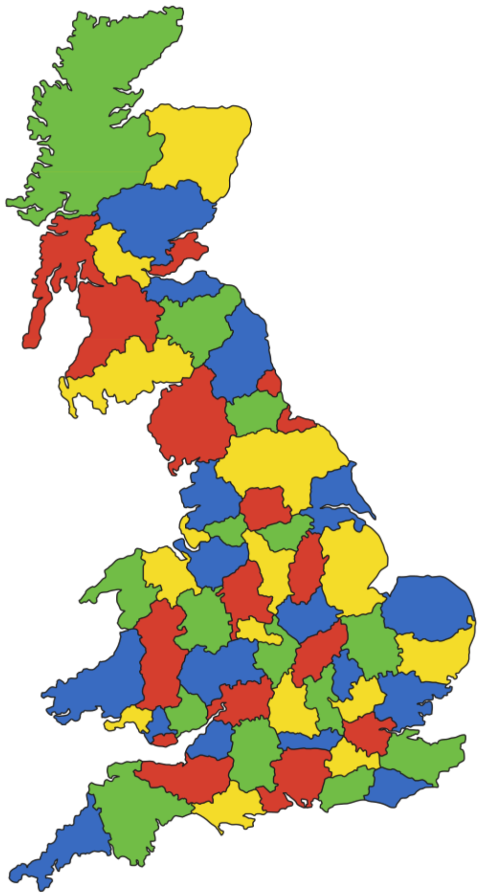
3. In all examples, $f_G(k)$ turned out to be a polynomial in k. What can you say about
   (a) its leading coefficient (i.e. the coefficient at the highest power of k);
   (b) its degree;
   (c) the second leading coefficient;
   (d) the constant term;
   (e) the lowest non-zero term;
   (f) anything else?

4. [Deletion-contraction] Pick an edge e in a graph G. Denote by $G-e$ the graph obtained from G by *deleting* e, and by G_e the graph obtained from G *contracting* e (i.e. identifying the vertices of e and removing multiple edges if needed). Find a relation between $f_G(k)$, $f_{G-e}(k)$, and $f_{G_e}(k)$.

5. Use this *deletion-contraction relation* to prove that $f_G(k)$ is a polynomial.

6. Use this relation to compute $f_{C_n}(k)$ for $n = 4, 5, \ldots, n$.

Planar graphs, Euler's formula, and Four Color Problem



Till Next Time!