De Bruijn sequences, chains of dominos, the bridges of Königsberg

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1 Warmups-1

Let's start together, I'll show you this trick. If you know right away how it works, don't spoil it quite yet! Let's break into small groups (4 - 6 per group) to discuss it, maybe perform it if you've figured out how it works. You can work on the "alternate" warmup puzzle, too.

1.1 First Warmup: Rock Paper Scissors Prediction Trick

I saw this trick on "Penn and Teller's Fool Us" – and it fooled them! But it might not fool math circle students. And maybe it's a little relevant to what we'll do.

I'll need three volunteers...

1.2 Alternate (off-topic) Warmup Problem

In how many ways can you write the digits 0 through 9 in a row, such that each digit other than the leftmost is within one of some digit to the left of it?

For example, here's one such sequence: 0 1 2 3 4 5 6 7 8 9.

And here's another: 3 4 5 2 6 1 0 7 8 9.

I'm interested in the answer – a little, at least – but even more interested in how one might start to think about the problem, and how you might explain your answer.

2 A Card Trick

I'm going to try to do this without pencil and paper, but I;m a little rusty and might need to use them to get it to work).

Anyway, the *mathematical* idea is still good! See if you can figure out how it works.

I'll need five volunteers...

2.1 After the puzzle

Can you identify the mathematical underpinnings that make this trick work?

3 Formal Definition and some questions

A *de Bruijn sequence of order n* is a sequence of letters (or numbers or other symbols) in which every possible sequence of length n of those letters occurs exactly once (the sequence is cyclic, the end of the sequences loops back to the beginning). So for example the sequence: 0110 is a de Bruijn sequence of order 2 for the symbols "0" and "1") since the first two digits are 01, the second two are 11, the third two are 10 and the fourth two (wrapping around) are 00.

(1) Can you find a de Bruijn sequence of order 3 for the symbols "0" and "1"?

(2) Can you find a de Bruijn sequence of order 2 for the symbols "0" and "1" and "2"?

(3) Can you find a 100 digit number (in base 10) in which every possible 2-digit sequence appears (again, allowing wrapping around – this is just asking for a de Bruijn sequence of order 2 for the digits 0–9)?

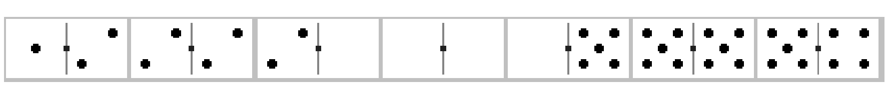
(4) In general, can you always find a de Bruijn sequence of any finite length on any finite set of symbols? If not, describe when you can and when you can't. (you do not need to solve this one to continue with the other questions – and we'll return to it)

(5) Any ideas on how we might generalize this or any related questions? For example, what would be the 2-dimensional version of a debruijn sequence? Or what if we wanted no repeating consecutive digits in the sequence? Can you think of any other restrictions we might want to try making on a sequence of digits?

4 Dominoes

Each 1×2 domino comes with a pair of numbers on it (with blank representing 0), the same pair occurs exactly once in a set of dominoes. "Double Sixes" dominoes use the numbers 0–6, "Double Nines" use the numbers 0–9, "Double Twelves" use the numbers 0–12. They don't sell them, but we could imagine a set of "Double Threes".

(6) How many dominoes are in a set of Double Sixes? Can you arrange them all in a line, in which adjacent dominoes have the same number on their adjacent sides? Can you do it so the last number on the left side agrees with the last number on the right side (i.e. make them into a loop)?



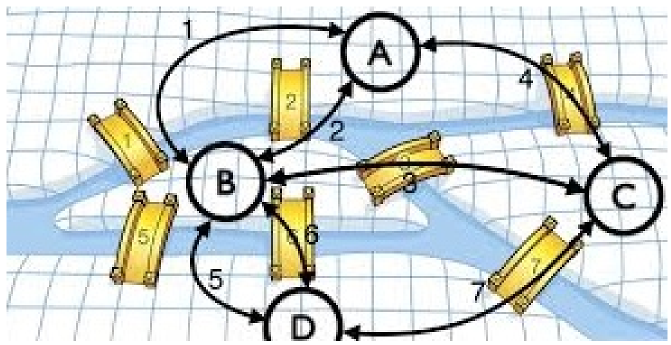
(7) How many dominoes are in a set of Double Nines? Can you arrange them all in a line, in which adjacent dominoes have the same number on their adjacent sides?

(8) If you *can't* do it, how close can you come? Can you use all but one domino? Or all but two?

5 Do you know the Bridges of Königsberg?

From Wikipedia:

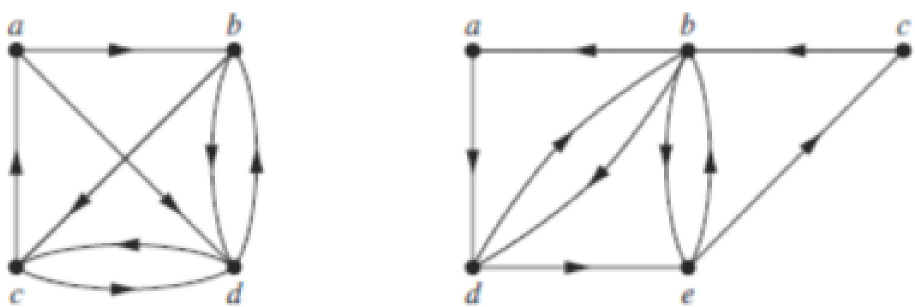
> The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands - Kneiphof and Lomse - which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.



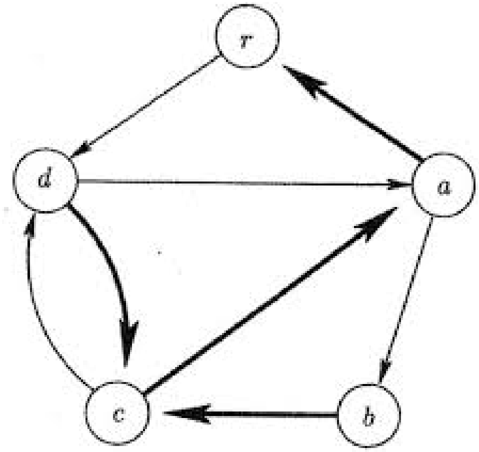
How can we find such a walk, or prove that no such walk exists? What might this have to do with the dommino or de Bruijn cycle problems?

5.1 What if the bridges were one-way?

In these examples, can you find a path through these graphs that uses every one-way "bridge" exactly once? Can you do it and end on the same node you began? Can we figure out necessary or sufficient criteria to be able to do it? and what does this have to do with the domino or de Bruijn cycle problems?



(a) (b)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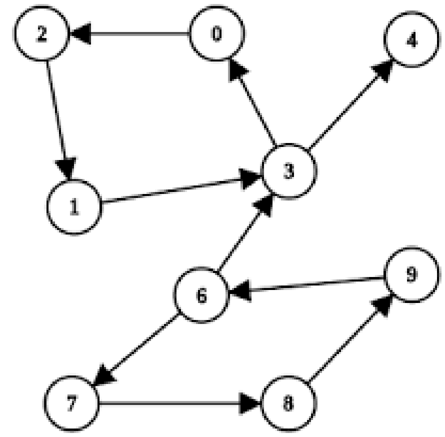
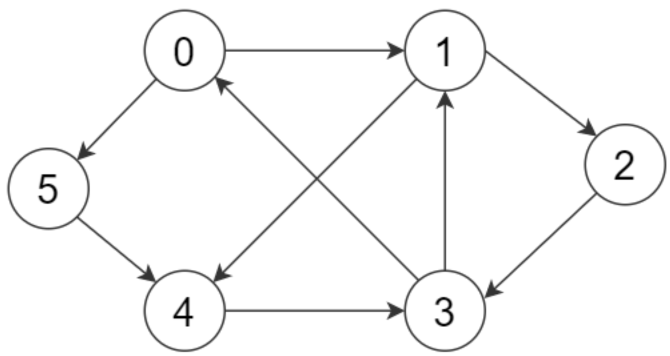

2
0
4
1
3
6
9
7
8


0
1
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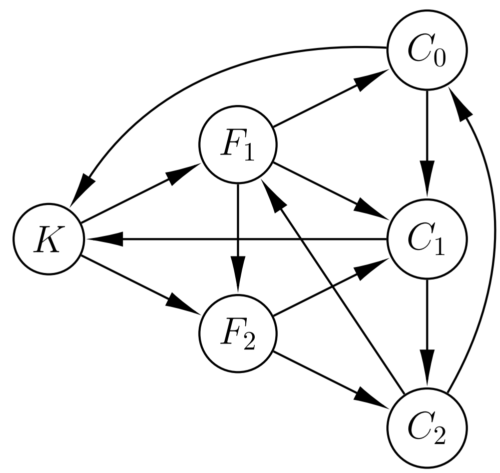
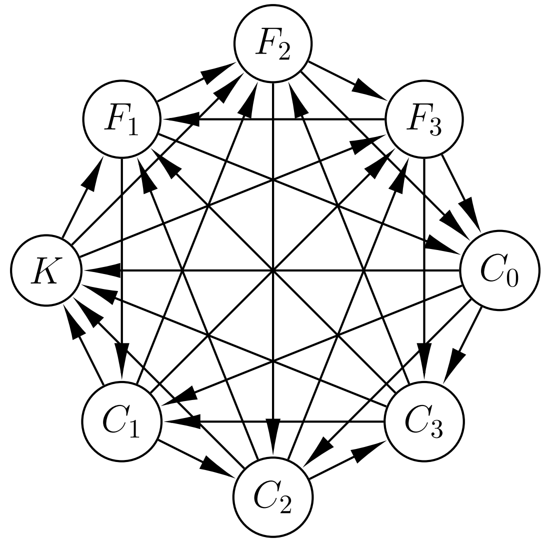
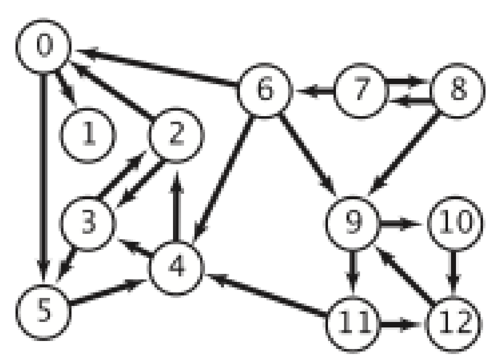

C_0
F_1
K
C_1
F_2
C_2


F_2
F_1
F_3
K
C_0
C_1
C_3
C_2


0
6
7
8
1
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10
4
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11
12

6 Graph Theory

A **graph** is a collection of *vertices* (aka points), and *edges* (often represented as lines or curves connecting the vertices). A **directed graph** has vertices, but instead of edges, we have *arrows* connecting the vertices, i.e. they specify a direction you may travel from one vertex to another.

These have many applications and interpretations in mathematical contects; they can be simple or complicated, some can drawn in the plane without the edges crossing, others can't. Some graphs are *extremely* complicated. (We can look at some examples online)

There are *many* interesting questions you can ask about graphs and directed graphs...

6.1 More definitions

- a **connected** graph
- degree of a vertex
- in- and out- degree of a vertex in a directed graph
- Eulerian path
- Eulerian circuit

6.2 The statement of the theorem

In a *connected* graph, there's an Eulerian path when at most two vertices have odd degree; there's an Eulerian circuit when every path has an even degree

In a *connected* directed graph, there's an Eulerian circuit when every vertexes out-degree is equal to its in-degree.

6.3 What does this have to do with deBruijn sequences or dominoes?

How can we represent the domino problem as an Eulerian path or circuit on a (directed) graph? What are the vertices, what are the edges?

How can we represent the Debruijn problem as an Eulerian path or circuit on a (directed) graph? What are the vertices, what are the edges?

6.4 some graphs