1 Billiards

Let $n, m$ be two coprime positive integers. Suppose a ball is launched from the bottom left corner of a $m \times n$ box at a $45^\circ$ angle and continually bounces on the sides until it reaches another corner, like below.

**Exercise 1.1.** Draw out the picture for some other pairs of $(m, n) = (3, 4), (2, 7), (2, 9), (6, 7)$. 
Exercise 1.2. If we label the bottom left as \((0, 0)\), which points are hit by the ball?

Definition 1.3. We label the bounces on the bottom with a positive sign if it goes left to right and a negative sign if it goes right to left. The pool sign \((m|n) = \pm 1\) is the product of the signs.

Exercise 1.4. Calculate the pool signs of the different pairs of \((m, n)\).

Exercise 1.5. What is a pattern for \((n - 1|n)\)?

Exercise 1.6. What is a pattern \((2|n)\)?

Exercise 1.7. Calculate \((m|n)\) for the following pairs: \((m, n) = (3, 8), (5, 8), (7, 8), (9, 8), (3, 10), (7, 10), (9, 10)\). Do you notice any pattern in the path the billiards ball takes?

Exercise 1.8. Determine a pattern for \((m|2n)\) where \(m\) is odd and \(2n\) is even.

2 Reciprocity

We want to determine if there is a relation between \((m|n)\) and \((n|m)\).

Exercise 2.1. How can we view \((n|m)\) and \((m|n)\) on the same board?

Exercise 2.2. Suppose that \(m, n\) are odd with \(m < n\). For \(0 < 2k < m\), show that the bounce at \((0, 2k)\) and \((n - m + 2k, 0)\) have the same sign.

Exercise 2.3. Show that if \(n, m\) are both odd, then \((n|m) \cdot (m|n) = (m|n - m)\).

Exercise 2.4. Prove a reciprocity that \((n|m)(m|n) = (-1)^{(m-1)(n-1)/4}\).