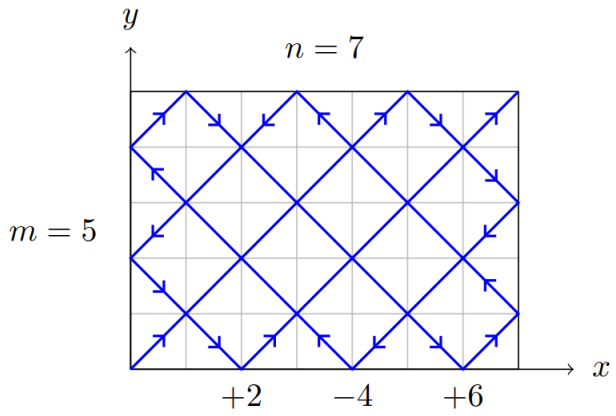
Billiards and Checkers I

BMC Advanced

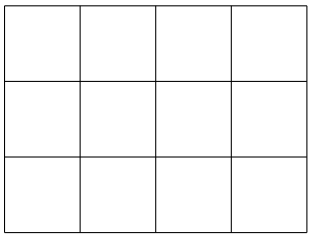
March 13, 2024

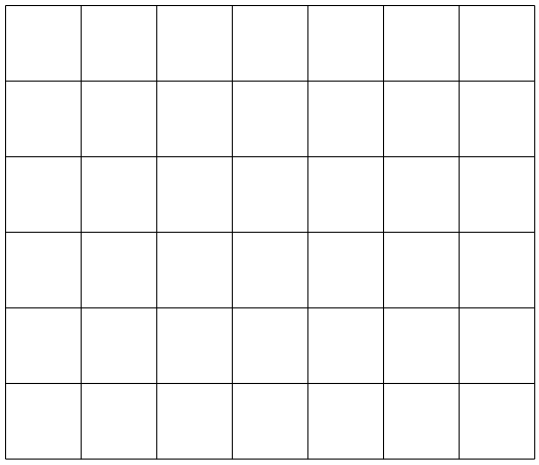
1 Billiards

Let n, m be two coprime positive integers. Suppose a ball is launched from the bottom left corner of a $m \times n$ box at a 45° angle and continually bounces on the sides until it reaches another corner, like below.



Exercise 1.1. *Draw out the picture for some other pairs of* $(m, n) = (3, 4), (2, 7), (2, 9), (6, 7)$.





Exercise 1.2. *If we label the bottom left as* $(0,0)$*, which points are hit by the ball?*

Definition 1.3. *We label the bounces on the bottom with a positive sign if it goes left to right and a negative sign if it goes right to left. The pool sign* $(m|n) = \pm 1$ *is the product of the signs.*

Exercise 1.4. *Calculate the pool signs of the different pairs of* (m,n)*.*

Exercise 1.5. *What is a pattern for* $(n-1|n)$*?*

Exercise 1.6. *What is a pattern* $(2|n)$*?*

Exercise 1.7. *Calculate* $(m|n)$ *for the following pairs:* $(m,n) = (3,8)$*,* $(5,8)$*,* $(7,8)$*,* $(9,8)$*,* $(3,10)$*,* $(7,10)$*,* $(9,10)$*. Do you notice any pattern in the path the billards ball takes?*

Exercise 1.8. *Determine a pattern for* $(m|2n)$ *where* m *is odd and* $2n$ *is even.*

2 Reciprocity

We want to determine if there is a relation between $(m|n)$ and $(n|m)$.

Exercise 2.1. *How can we view* $(n|m)$ *and* $(m|n)$ *on the same board?*

Exercise 2.2. *Suppose that* m,n *are odd with* $m<n$*. For* $0<2k<m$*, show that the bounce at* $(0,2k)$ *and* $(n-m+2k,0)$ *have the same sign.*

Exercise 2.3. *Show that if* n,m *are both odd, then* $(n|m)\cdot(m|n) = (m|n-m)$*.*

Exercise 2.4. *Prove a reciprocity that* $(n|m)(m|n) = (-1)^{(m-1)(n-1)/4}$*.*