Worksheet

1. The upper surface of a one-dimensional bowl is described by the function

$$y(x) = -ax^4 + bx^2 + c$$

where a, b, c are constants and $a, b > 0$. A ball of mass m is released from rest in the bowl and allowed to freely oscillate.

   (a) The only force acting on the ball is gravity, which has the potential energy relation

   $$U(y) = mgy$$

   where m is the mass of the ball, $g = 9.8\,m/s^2$ is the acceleration due to gravity, and y is the vertical position of the ball. What is the gravitational potential energy at each point on the surface of the bowl? Write it out in terms of the horizontal position x.

   (b) At what x-position is the potential energy minimized? Only consider positions which are local minima. This is because the function $y(x)$ goes to negative infinity to the left and right of the bowl's local minimum, but let's assume the ball doesn't leave the bowl. In other words, we only want *stable equilibria*, i.e. minima where $U''(x) > 0$.

   (c) Show that, without any approximations, the equation of motion for the ball is

   $$\frac{d^2x}{dt^2} = -2gbx + 4gax^3$$

   (d) This equation is nonlinear in x and quite hard to solve. Show that if we assume the ball is released at a small distance $x = x_0$ from the potential energy minimum, the ball's motion is described by simple harmonic motion. i.e. the equation of motion can be approximated as

   $$\frac{d^2x}{dt^2} = -\omega^2 x$$

   where ω is a constant you find. [Hint: find the Taylor approximation of $U(x)$ up to the quadratic term, *OR* Taylor approximate the RHS of the equation of motion to the linear term].

   (e) Find the solution of the differential equation, $x = x(t)$, assuming the ball is released at $t = 0$ at a position $x = x_0$ and with zero initial velocity.

   (f) Using the value found for the angular frequency ω, write down the period of the ball's motion (how long a full oscillation takes to complete).

   (g) For certain values of a and b, our approximation is very accurate. Note the approximation doesn't change at all with c, why is this? Either qualitatively or quantitatively explain for what values a and b our approximation is no longer accurate.