Worksheet solutions

1. The upper surface of a one-dimensional bowl is described by the function

$$y(x) = -ax^4 + bx^2 + c$$

where a, b, c are constants and $a, b > 0$. A ball of mass m is released from rest in the bowl and allowed to freely oscillate.

(a) The only force acting on the ball is gravity, which has the potential energy relation

$$U(y) = mgy$$

where m is the mass of the ball, $g = 9.8\,m/s^2$ is the acceleration due to gravity, and y is the vertical position of the ball. What is the gravitational potential energy at each point on the surface of the bowl? Write it out in terms of the horizontal position x.

Solution: Plugging $y(x)$ into $U(y)$ we get the following gravitational potential energy for each location x on the bowl

$$U(x) = U(y(x)) = -mgax^4 + mgbx^2 + mgc$$

(b) At what x-position is the potential energy minimized? Only consider positions which are local minima. This is because the function $y(x)$ goes to negative infinity to the left and right of the bowl's local minimum, but let's assume the ball doesn't leave the bowl. In other words, we only want *stable equilibria*, i.e. minima where $U''(x) > 0$.

Solution: We first want to find all local extrema of $U(x)$. To do this, we find all points x where $U'(x) = 0$

$$U'(x) = 0 = -4mgax^3 + 2mgbx = 2mgx(b - 2ax^2)$$

So $x = 0, +\sqrt{b/2a}, -\sqrt{b/2a}$ are our local extrema. To find if they are minima or maxima, we can use the second derivative test:

$$\begin{aligned} U''(x) &= -12mgax^2 + 2mgb \\ U''(0) &= 2mbg > 0 \\ U''\left(\sqrt{b/2a}\right) = U''\left(-\sqrt{b/2a}\right) &= -6mgb + 2mbg = -4mgb < 0 \end{aligned}$$

So $x = 0$ corresponds with a local minimum and $x = \pm\sqrt{b}2a$ corresponds with local maxima. Therefore $x = 0$ is the point where the potential energy is minimized (within the bounds of the bowl).

(c) Show that, without any approximations, the equation of motion for the ball is

$$\frac{d^2x}{dt^2} = -2gbx + 4gax^3$$

Solution: The equation of motion is given by

$$m\frac{d^2x}{dt^2} = F = -\frac{dU}{dx}$$

Where $U'(x)$ was found in (b) can can simply by plugged in:

$$m\frac{d^2x}{dt^2} = 4mgax^3 - 2mgbx \qquad \rightarrow \qquad \frac{d^2x}{dt^2} = -2gbx + 4gax^3$$

(d) This equation is nonlinear in x and quite hard to solve. Show that if we assume the ball is released at a small distance $x = x_0$ from the potential energy minimum, the ball's motion is described by simple harmonic motion. i.e. the equation of motion can be approximated as

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where k is a constant you find. [Hint: find the Taylor approximation of $U(x)$ up to the quadratic term, *OR* Taylor approximate the RHS of the equation of motion to the linear term].

Solution: The exact potential energy has the form $U(x) = -mgax^4 + mgbx^2 + mgc$. Approximating this up to second order, i.e. up to a quadratic term, will yield the same potential as a simple harmonic oscillator, as shown in lecture. We Taylor expand about the minimum $x = 0$ ...

$$U(x) \approx U(0) + U'(0)x + \frac{1}{2}U''(0)x^2$$

Since $x = 0$ is a local minimum, $U'(0) = 0$, and in part (b) we found $U''(0) = 2mbg$. Substituting these in, we find

$$U(x) \approx mbgx^2 + c \qquad \rightarrow \qquad F = -2mbgx \qquad \rightarrow \qquad \frac{d^2x}{dt^2} = -2gbx$$

So our approximation indeed yields simple harmonic oscillation! And the angular frequency of this oscillation is $\omega^2 = 2gb$.

(e) Find the solution of the differential equation, $x = x(t)$, assuming the ball is released at $t = 0$ at a position $x = x_0$ and with zero initial velocity.

Solution: The general solution for simple harmonic motion as found in lecture was

$$x(t) = A\cos(\omega t + \delta)$$

In this case we found $\omega = \sqrt{2bg}$ and A and δ can be found by plugging in the given intial conditions:

$$v(0) = x'(0) = -A\omega\sin(0 + \delta) \qquad \rightarrow \qquad \delta = 0$$
$$x(0) = x_0 = A\cos(0 + \delta) \qquad \rightarrow \qquad A = x_0$$

Therefore the solved equation of motion of the ball with the given initial conditions is

$$x(t) = x_0\cos(t\sqrt{(2gb)})$$

(f) Using the value found for the angular frequency ω, write down the period of the ball's motion (how long a full oscillation takes to complete).

Solution: The period of motion is related to angular frequency ω by the following formula:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2gb}}$$

(g) For certain values of a and b, our approximation is very accurate. Note the approximation doesn't change at all with c, why is this? Either qualitatively or quantitatively explain for what values a and b our approximation is no longer accurate.

Solution: The quality of the simple harmonic oscillator approximation is not dependent on c since it simply shifts the potential energy up and down, and doesn't change the shape/derivatives of the curve. At a deeper level, potential energy distributions are always defined somewhat arbitrarily in physics because they can all be shifted by some constant factor and the equation of motion will not change. This stems from the fact that the force on objects follows $F = -\frac{dU}{dx}$ so constant terms in the $U(x)$ vanish.

Qualitatively, the quality of our approximation is improved when the potential well more closely resembles a parabola. One way to quantify this is to look at the higher-order terms in the Taylor expansion:

$$U(x) \approx U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \frac{1}{6}U'''(0)x^3 + \cdots + \frac{1}{n!}U^{(n)}x^n$$

If the coefficients of the higher-order terms are small, then their contribution to $U(x)$ relative to the x^2 will be small, and therefore we will have a better approximation. A concrete way to do this is to compare the coefficients of the x^2 term, and the next highest term, x^3. We want

$$\frac{1}{2}U''(0)x^2 \gg \frac{1}{6}U'''(0)x^3 \qquad \rightarrow \qquad x \ll \frac{U''(0)}{U'''(0)}$$

This is satisfied when we have very small oscillations in x, i.e. the LHS is very small, and when the $U''(0)$ is much larger than the $U'''(0)$ term, i.e. the potential looks more parabolic. In our case, $U''(0) = 2mgb$ and $U'''(0) = 0$ so we do expect to see a very good approximation in this case, no matter our choice of a and b.