Quadratic irrationalities

1. Find the 100th digit after the point in the decimal expansion of $(3+\sqrt{5})^{2023}$.

2. Find the largest power of 2 which divides $\lfloor(1+\sqrt{3})^{99}\rfloor$.
   (Here $\lfloor x\rfloor$ is the integer part of number x).

3. Find the last digit of the number $\lfloor(\sqrt{2}+\sqrt{3})^{100}\rfloor$.

4. Show that the equation

$$(x+y\sqrt{2})^4+(z+t\sqrt{2})^4=7+5\sqrt{2}$$

   has no solutions in rational numbers x, y, z, and t.

5. Show that $(1+\sqrt{2})^{2023}$ can be written as $\sqrt{N}+\sqrt{N+1}$ for some integer N.

6. Find the value of the *continued fractions*: $1+\cfrac{1}{2+\cfrac{1}{2+\cdots}}$

7. Show that the equation $|x^2-2y^2|=1$ has infinitely many positive integer solutions. Can you find all solutions? Can you find a connection with the previous problem?

8. (a) The number 36 is both a *square* and a *triangular* number:

$$36=6^2=1+2+3+4+5+6+7+8.$$

   Find two more such numbers.

   (b) Show that there are infinitely many such numbers.

   (c) Find all of them.