Berkeley Math Circle **January 2023**
Probability and Random Variables **Instructor: Paul Zeitz**
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Problem Set I

1 What's the probability of tossing a fair coin 10 times and seeing

(a) No heads?

(b) At least 1 head?

(c) Exactly 5 heads?

2 What's the probability of rolling six dice and

(a) getting a sum of 6?
(b) getting a sum of 7?
(c) getting a sum of 10?
(d) having all six numbers be equal?
(e) having all six numbers be different?

3 Two cards are placed in a hat. One card is red on one side, and black on the other side. The other card is black on both sides. The hat is shaken, and someone draws a card, showing you one side. It is black. What is the probability that the other side is black?

4 There are three coins in a box. One is a two-headed coin; another is a fair coin; and the third is a biased coin that comes up heads 75% of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

5 Suppose a test for a disease has a 99% accuracy probability; i.e., there is a 1% chance that the test gives the wrong answer. Suppose that the probability of actually having the disease is is 0.5%. Now suppose you take the test, and you get a "positive" result. What is the probability that you actually have the disease?

6 *Face Cards*. A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a "face card;" i.e., a Jack, Queen, or King? Do you want to take this bet?

7 *First Ace*. Shuffle a standard deck of 52 cards and draw cards until you see the first ace. Let F be the position of this first ace. Thus $1 \leq F \leq 49$. What is the most likely value for F?

8 Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

(a) the game never ends?

(b) the first player wins?

(c) the second player wins?

9 *The Classic Birthday Problem.* In a room of 30 people, how likely is it that at least two people have the same birthday?

10 *Another Birthday Problem.* In Klopstockia, all factory workers must work every day of the year, getting no days off, with only one exception: whenever it is a worker's birthday, the factory closes down and everyone gets the day off! Factories are obligated to hire workers randomly, without discriminating on the basis of birthday. In order to maximize output, how many workers should a factory employ?

11 *The Classic Gambler's Ruin Problem.* Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars, and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e., has zero dollars) the game is over. What is the probability that A goes bankrupt?

What happens if the probabilities are not equal; i.e., what if the probability that the coin is heads is p, for some fixed $0 \leq p \leq 1$.

12 *What a Loser!* You arrive in Las Vegas with \$100 and decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money. Which of the following strategies is best? Or are they all equally bad?

(a) Making bets of \$1 each time.

(b) Making bets of \$10 each time.

(c) Making a single bet of \$100.

13 It costs a consumer \$1 to buy a Klopstockia lottery ticket. The buyer then scratches the ticket to see the prize. Compute, to the nearest penny, the expected profit that the state of Klopstockia makes per ticket sold, given the following scenarios for prizes awarded. (The state will make a profit if the expected value of the lottery ticket is *less* than \$1.)

(a)

Prize	\$1	\$10
Probability	$\frac{1}{10}$	$\frac{1}{1,000}$

(b)

Prize	\$1	\$10	a free lottery ticket
Probability	$\frac{1}{10}$	$\frac{1}{1,000}$	$\frac{1}{5}$

14 (a) On average, how many times must a die be thrown until one gets a 6?

(b) How many times, on average, should one toss a fair die in order to see all 6 possible outcomes?

15 What is the expected value of the random variable F of Problem **??**?

16 *A Gambling "System."* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$ depending on whether you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. You are guaranteed to make a profit? Right? Use expectation to show that this won't work. What if you triple instead of double?

17 *The St. Petersburg Paradox.* Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I'll pay you $2. If it takes two flips, then I'll pay you $4. Three flips, $8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

How much is this game worth *to you*? In other words, if a there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you'd pay at least 1 dollar. In fact, you'd almost certainly pay 2 dollars. How about 3? 4? 5? More?

18 *A true story.* When I first organized the Bay Area Mathematical Olympiad, I needed to send registration forms out with random ID numbers for participants. So I made a list of the numbers from 1 to 1000, and then used my sampling software to take a random sample of size 1000 from these numbers. However, I stupidly forgot to check the "sample without replacement" button and instead I sampled *with* replacement. How many distinct ID numbers were produced?

19 *Snake necklaces.* Imagine a pit of 100 snakes, and James Bond is thrown into the pit. He fearlessly, and randomly, grabs ends of snakes (ignoring whether it is the head or tail) and deftly ties them together. He keeps doing this until he is left with a bunch of "snake necklaces" that cannot harm him. How many necklaces will there be?

20 *A true story.* In the SF Math Circle for elementary school kids, 11 8-year-old kids stood in a circle. They wrote their names on a piece of paper, and the instructor put them in a box and shook the box. Then each kid randomly chose a name. The instructor handed a kid a beanbag ball and the kid then tossed the ball to the person whose name they had. And so it continued. If not all kids got a ball tossed to them, the instructor gave the ball to one of those left-out kids and the process continued.

(a) If a kid ended up tossing a ball to him or herself, that kid cried. On average, how many kids cry?

(b) What is the probability that no kids cry?

(c) The instructor wanted all the kids to be able to toss the ball without intervention. In other words, ideally, all 11 kids will form a "cycle." What is the probability that this happens?

(d) If all the kids are not in one cycle, the instructor asked the kids to change names so that this can be achieved. To keep anarchy at bay, the instructor only allowed two kids at a time to exchange their slips of paper. On average, how many such exchanges are needed?

(e) Another desirable scenario for the instructor was for a majority of the kids to be in a cycle. Otherwise, kids have tantrums. What's the probability of a tantrum?