Intro to Formal Logic

November 29 & December 6, 2023

Note: This is the first of a two-part lecture series on Formal Logic--this half will give some basic intuition while the second part will formalize some logical notions.

Section 0: What is Logic?

You're probably familiar with some form of logic, either in mathematics or else in your daily life. Effectively, logic is how you get from one statement to another. For instance, you might know that if a number is an integer, it is odd or even, and that if an integer is even and divisible by 3, it is divisible by 6. Thus, you could conclude, using logic, that if an integer is divisible by 3 but not by 6, it is odd.

Another example could occur if you are going to Berkeley Math Circle, know that the Intermediate I session meets in a classroom in Cheit, and know that none of the classrooms in Cheit have more than 100 seats. In this case, if you found yourself in a classroom with more than 100 seats, you could logically conclude that you are not in the right classroom for Intermediate I.

In both of these cases, we had some number of premises (for instance, Intermediate I meets in Cheit, and no classrooms in Cheit have more than 100 seats), and concluded a new statement (if you're in a classroom with more than 100 seats, you are not in the Intermediate I classroom) from those premises. In general, this is the backbone of logic: using certain statements we take as true (premises) to derive other true statements (conclusions).

To begin our discussion, let's take a look at a word we very often encounter when discussing logic.

Section 1: “if” and “iff”

Consider the following statement: if an integer is divisible by 4, it is divisible by 2. I think we’d all agree with this statement, but what does it actually mean?

We all have a decent intuition for the word “if,” but it’s somewhat difficult to define without just saying “if” again. Looking at the above example, we might try to figure out what sort of combinations of divisibility by 2 and 4 an integer could have. In this case, we may note four distinct possibilities:

-An integer could be divisible by 2 and 4.
-An integer could be divisible by 2 but not 4.
-An integer “could” be divisible by 4 but not 2.
-An integer could be divisible by neither 2 nor 4.

In this case, we see that all but the third case is possible. This makes sense; if we had an integer divisible by 4 but not 2, it would be a counterexample and the statement “if an integer is divisible by 4, it is divisible by 2” would not be true.

In a similar vein, consider the statement “if it is Thursday, Berkeley Math Circle will not be meeting.” What this statement means is that, given any day, either

-That day is Thursday and BMC is not meeting.
-That day is not Thursday and BMC is meeting.
-That day is not Thursday and BMC is not meeting.

But it is never Thursday while BMC meets.

This motivates the following definition:

Definition 1.1: Let p and q be statements, such as “BMC is not meeting on this day” or “x is divisible by 2.” Then, given that it is impossible for both p to be true and q to be false at the same time, we may say that “if p, then q.”

Definition 1.2: Let p and q be statements, such as “BMC is not meeting on this day” or “x is divisible by 2.” If we find some example that satisfies p but not q, then we may call that a *counterexample* to the statement “if p, then q.”

We may abbreviate “if p, then q” as “p→q,” and we may sometimes say “p implies q” instead, which has the exact same meaning.

We can summarize our results in a truth table, which you may write below:

“If” statements can be tricky. For instance, the statement “if an integer is odd and divisible by 6, it is raining in New York City” is technically true. Why? Well, all that we need for an “if” statement to hold is that whenever the first part is true, so is the second. So if the first part of the statement is never true, the “if” statement is known as *vacuously* true, because we’ll never find a counterexample.

Let “p→q” be an “if” statement. Then we may call “q→p” its converse.

Exercise 1.1) Does an “if” statement always imply its converse? Why or why not?

Definition 1.3: If the statements “p→q” and “q→p,” both hold, we may say that “if and only if p, then q.”

We frequently abbreviate “if and only if p, then q” as “iff p, then q,” (with two “f”s), and we may also write it as “p↔q.”

Exercise 1.2) Write a truth table for “p↔q.”

We might notice that we can chain together "if" statements, as we did in part 1. For instance, we know that if x is divisible by 72, then x is divisible by 9, and if x is divisible by 9, its digit sum is divisible by 9 (if you didn't know this before, it's true, and very cool. I don't have time for the proof here, but I highly recommend you look it up). Therefore, we can conclude that if x is divisible by 72, its digit sum is divisible by 9.

But what exactly is going on here, logically? Well, to clean things up, we'll let "p" be the statement that x is divisible by 72, "q" be the statement that x is divisible by 9, and "r" be the statement that the digit sum of x is divisible by 9. Then what we have asserted is "p→q" and "q→r," and what we have claimed that we can conclude is "p→r." But why is this justified?

Exercise 1.3) Using truth tables, show that from "p→q" and "q→r" we may conclude "p→r."

Exercise 1.4) Using truth tables or otherwise, show that from "p↔q" and "q↔r" we may conclude "p↔r."

Exercise 1.5) Show that from "p→q" and "r→s" we may conclude "p and r→q and s."

Exercise 1.6) Show that for all propositions p and q, "(p→q)↔(not q→not p)." The statement "not q→not p" is called the contrapositive of "p→q," and is often very useful.

Section 2: What's in a Proof?

Most math is done through proofs, which are essentially a series of logical steps from a set of premises to a desired conclusion. For instance, let's make a proof that if a prime integer is greater than two, it is odd, using our rules for "if" statements that we have figured out before. We start with a few premises:

1) If an integer is prime, it is not divisible by any positive integers except 1 and itself.
2) If an integer is greater than 2, it is not equal to 2.
3) If an integer is not divisible by any positive integers except 1 and itself and not equal to 2, it is not divisible by 2.
4) If an integer is not divisible by 2, it is odd.

Well, we can use our result from exercise 1.5 to combine 1) and 2), concluding that if an integer is prime and greater than 2, then it is not divisible by any positive integers except 1 and itself and not equal to 2. Then we could use exercise 1.3 on this new statement and 3) to find that if an integer is prime and greater than 2, it is not divisible by 2. We could finally use exercise 1.3 on this and 4) to find that if an integer is prime and greater than 2, it is odd, which is what we were trying to prove.

This is very slow and clunky, and in general you won't have to write out all the steps like this (in a case like this, you could even just write out the premises and let the reader work out the logic going on behind it). But this kind of logic is fundamentally what is going on behind every proof you might write or encounter. In a very real sense, truth in mathematics is defined by these sorts of formal proofs.

Another thing that might come to mind when reading a proof like this is what exactly we are allowed to use as our "premises." In this case, they are mostly just definitions, but we will pay more attention to these "premises" in the next lecture. For now, just think of them as basic results, definitions, or assumptions that we know are true.

The proof we just did was essentially a chain of "if" statements that ended up eventually giving us the "if" statement we were looking for. Another common (and important) form of proof is *proof by contradiction*. To understand how this works on a logical level, we're going to need a new symbol.

Definition 2.1: If a statement p is false, we may say that "not p" is true.

We sometimes write "not p" as "$\neg p$."

In a proof by contradiction, we are trying to prove a statement of the form "$p \rightarrow q$." To do this, we start by assuming that p and $\neg q$ are both true. We then show that these assumptions lead to a statement being both true and false (i.e. there is some statement r such that r and $\neg r$ both hold), which is a contradiction. Since we are assuming that math does not have any contradictions, we can use the contrapositive to show that p and $\neg q$ can never both be true. However, this is just the definition of $p \rightarrow q$, so we are done.

This is all very theoretical, so let's try an example. We will go through this one together, first informally, then second by logically justifying every step of the way.

Exercise 2.1) Use a proof by contradiction to show that the square root of 2 is irrational; that is, that if x^2=2, then x cannot be written in the form p/q, where p and q are integers.

Section 3: Notation Practice

This section will just give a little more notation to help familiarize you with some things we will look at more next week. We will start with a little glossary of notation, some old, some new, then practice using them in some elementary proofs.

If-then/implies: $\rightarrow$
If and only if: $\leftrightarrow$
And: $\wedge$
Or: $\vee$
Not: $\neg$

For these exercises, p, q, r, and s are all statements

3.1) Give a complete proof that $((p\rightarrow q)\wedge(q\rightarrow r)\wedge(r\rightarrow p))\leftrightarrow((p\leftrightarrow q)\wedge(q\leftrightarrow r)\wedge(r\leftrightarrow p))$ always holds. Try it using previous results, and with truth tables.
3.2) Show that $((p\vee q)\wedge(p\rightarrow r)\wedge(q\rightarrow r))\rightarrow r$.
3.3) Show that $(((p\wedge q)\vee(r\wedge\neg p))\wedge(r\leftrightarrow p))\rightarrow(q\wedge r)$.
3.4) Show that $(((p\wedge\neg q)\rightarrow r)\wedge\neg r)\rightarrow(p\rightarrow q)$. This is proof by contradiction.