Harmonic Oscillators

Why *everything* is a spring (approximately)

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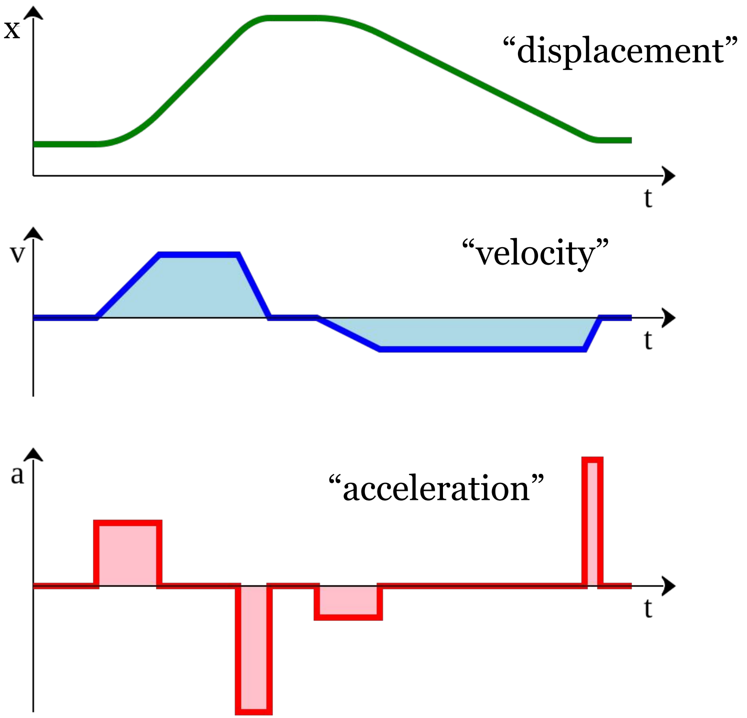
Outline

- What is a spring (mathematically)
  - Force, Newton’s second law
  - Hooke’s law
  - Linear homogenous second order ODEs
- Simple harmonic oscillator solution
  - Ansatz
    - Sin and cos
    - Complex numbers
  - Initial conditions
- BREAK
- Everything is a spring
  - Quadratic potential
  - Complex potential energy distributions
  - Taylor expansions
- Examples
  - Pendulum
  - Molecules

Force and Newton's second law

$$\boxed{F = ma}$$ (Newton's Second Law)

Force = Mass * Acceleration



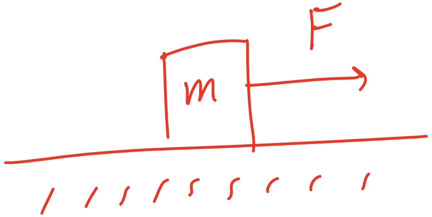
$x = x(t)$

$v = v(t)$

$a = a(t)$

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$



$$a = \frac{F}{m}$$

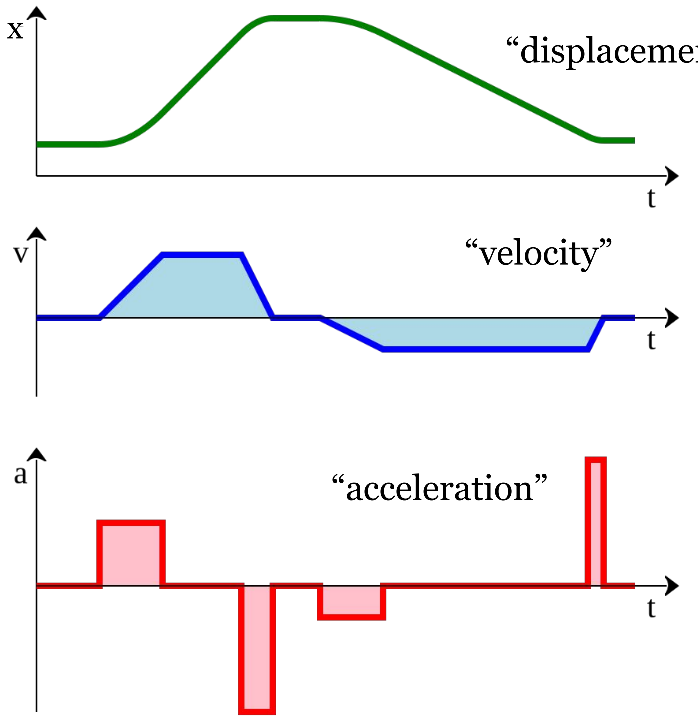
$$\frac{d^2x}{dt^2} = \frac{F}{m}$$ "eqn of motion"

In general we can solve this for $x = x(t)$, but F can be a function of many things...

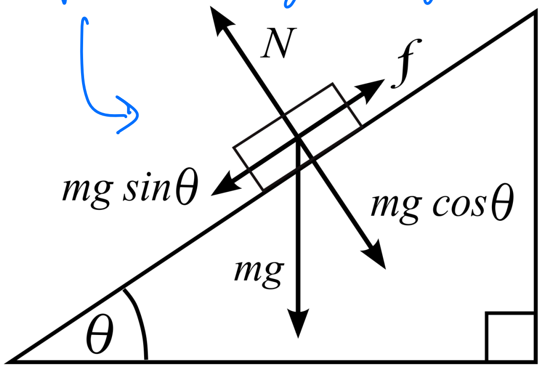
Force and Newton's second law

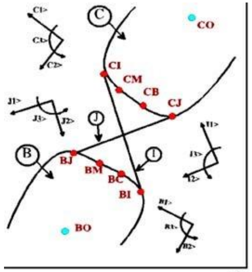
$$\boxed{F = ma}$$ (Newton's Second Law)

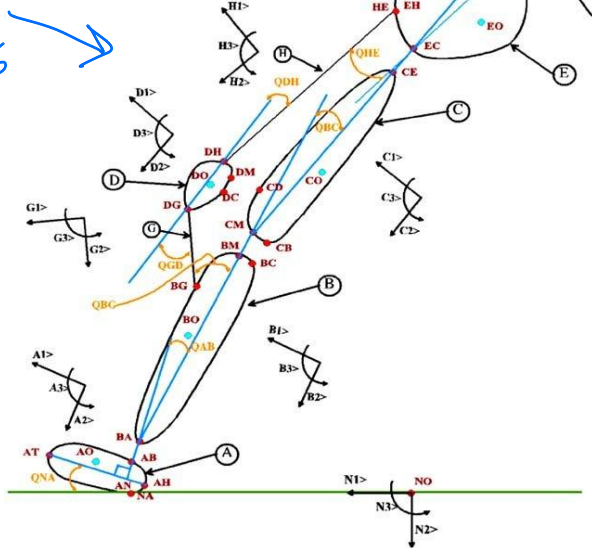
Force = Mass * Acceleration



If interested, F = ma usually set up by drawing free-body diagrams







Hooke's Law

- Linear relationship!
- True for *almost* all springs
- k is "strength" of spring

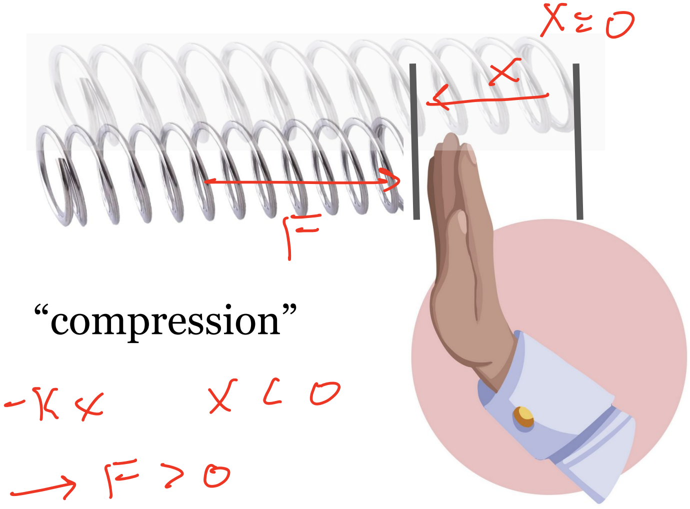
$$F = -kx$$

(Newton's Second Law)

Force = Spring constant * Displacement of spring from relaxed position

$k > 0$

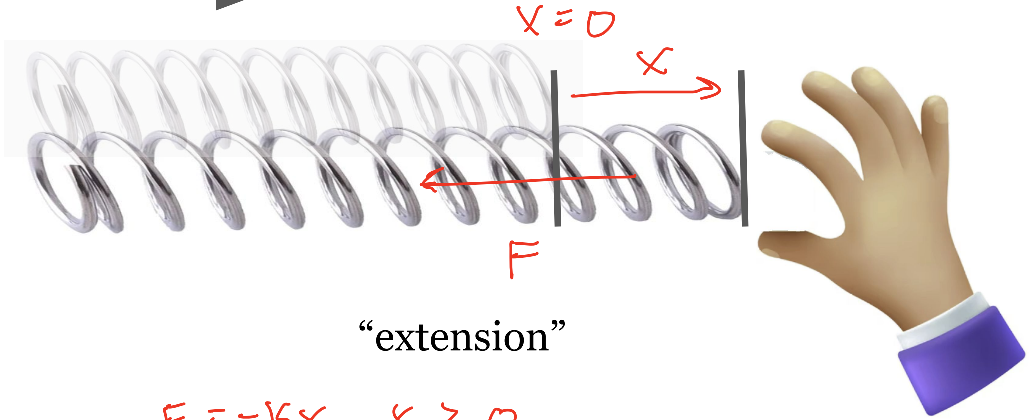
+x direction



"compression"

$F = -kx \quad x < 0$

$\rightarrow F > 0$



"extension"

$F = -kx \quad x > 0$

$F < 0$

Combining Newton's and Hooke's Laws

$$F = ma = m\frac{d^2x}{dt^2}$$

$$F = -kx$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

The spring "equation of motion"

This is a ...

Combining Newton's and Hooke's Laws

$$F = ma$$
$$F = -kx$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

The spring "equation of motion"

This is a ...

"Linear Homogenous Second Order Ordinary Differential Equation"

linear = highest power of x is 1.

homogenous = linear in the way derivatives/variables are combined

second order = highest derivative power is 2. $\frac{d^2x}{dt^2}$

ordinary = only 1 independent variable $x = x(t)$.

diff eq = independent variable is stuck in derivative!

→ how do we solve for it?

Solving the equation

we need an "ansatz" or "guess"

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

The spring "equation of motion"

→ what are functions that when you differentiate twice, you get back? (up to some constants)

trig functions!

Guess: $x(t) = A\cos(\omega t + \delta)$

$\frac{dx}{dt} = -A\omega\sin(\omega t + \delta)$

$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \delta)$

plug in → $-A\omega^2\cancel{\cos(\omega t + \delta)} = -\frac{kA}{m}\cancel{\cos(\omega t + \delta)}$

$A\omega^2 = \frac{kA}{m}$

$\omega = \sqrt{\frac{k}{m}}$

"angular frequency"

$\therefore x(t) = A\cos\left(\sqrt{\frac{k}{m}}t + \delta\right)$

Solution using complex numbers

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \rightarrow \frac{d^2z}{dt^2} = -\frac{k}{m} z$$

The spring "equation of motion"

Another ansatz:
convert the eqn to
complex numbers $z = x + iy$:

$$\frac{d^2z}{dt^2} = -\frac{k}{m} z$$

and guess

$$z = Ae^{i\omega t}$$

$$\frac{dz}{dt} = i\omega A e^{i\omega t}$$

$$\frac{d^2z}{dt^2} = -\omega^2 A e^{i\omega t}$$

$\downarrow$ plug in

$$-\omega^2 \cancel{Ae^{i\omega t}} = -\frac{k}{m} \cancel{Ae^{i\omega t}}$$

$$\omega^2 = \frac{k}{m} \rightarrow \omega = \sqrt{\frac{k}{m}}$$

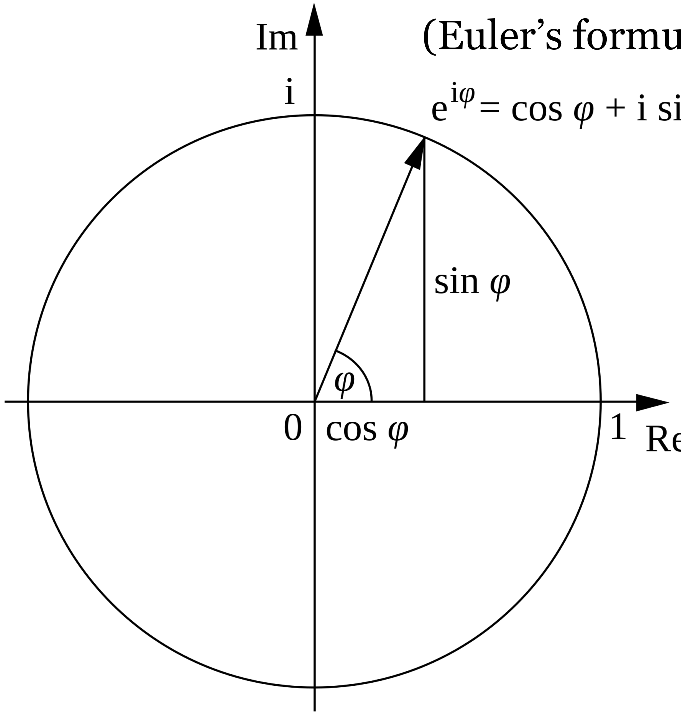
sol: $z = Ae^{i\omega t} = Ae^{i\sqrt{\frac{k}{m}}t}$?

$$\mathrm{Re}[z] = B\cos\left(\sqrt{\tfrac{k}{m}}t\right)$$

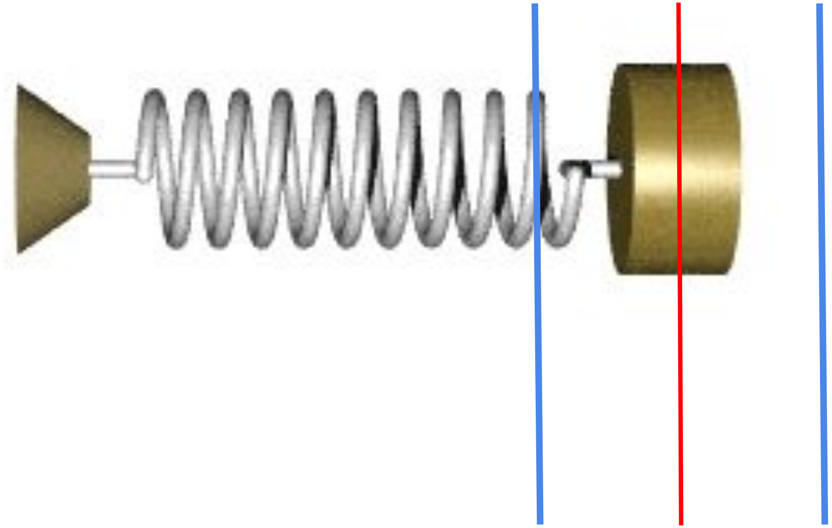
$$\mathrm{Im}[z] = C\sin\left(\sqrt{\tfrac{k}{m}}t\right)$$

(Euler's formula)

$e^{i\varphi} = \cos\varphi + i\sin\varphi$



Initial conditions



$$x(t) = \underline{A} \cos\left(\sqrt{\frac{k}{m}}\, t + \underline{\delta}\right)$$

what are these constants!

"the solution of an n-th order linear diff eq, will have n unknown variables"

These are found through "initial conditions":

1. position x_0 at a time t_0
2. velocity v_0 at a time t_0

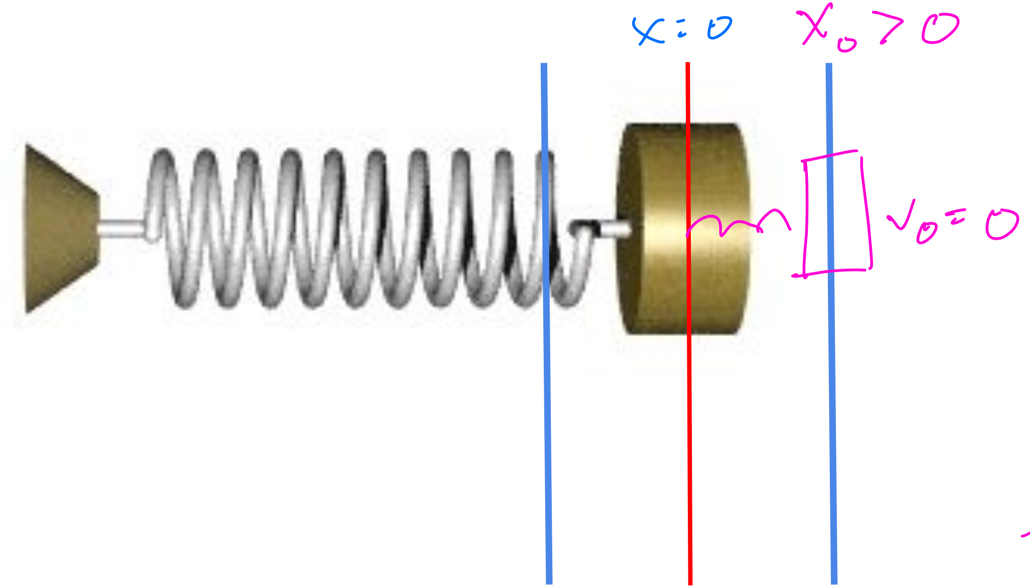
usually starting position and velocity

$$\begin{cases} x(t_0) = x_0 = A\cos\left(\sqrt{\frac{k}{m}}\, t_0 + \delta\right) \\ v(t_0) = v_0 = -\sqrt{\frac{k}{m}}\, A \sin\left(\sqrt{\frac{k}{m}}\, t_0 + \delta\right) \end{cases}$$

↳ 2 eqns, 2 unknowns

can solve for A, δ.

Initial conditions



Example solution:

Let the spring start at $x_0>0$ with velocity $v_0=0$ at $t=0$.

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \delta\right)$$

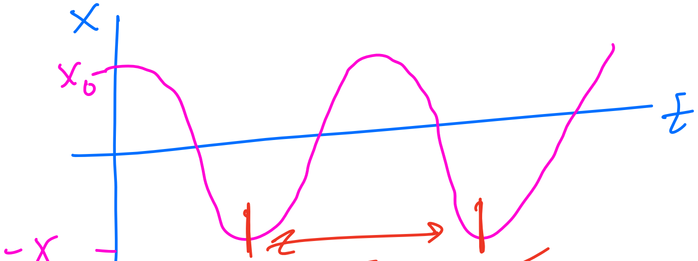
$$x'(t) = -\sqrt{\frac{k}{m}}\,A\sin\left(\sqrt{\frac{k}{m}}\,t + \delta\right)$$

$$x(0) = x_0 = A\cos\delta \rightarrow A = x_0$$

$$x'(0) = 0 = \sqrt{\frac{k}{m}}\,A\sin(\delta) \rightarrow \delta = 0 \text{ or } \cancel{A=0}$$

Solved eqn of motion!

$$x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$



$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Break timeeeee

Break timeeeee

Why *everything* is a spring (approximately)

Potential energy

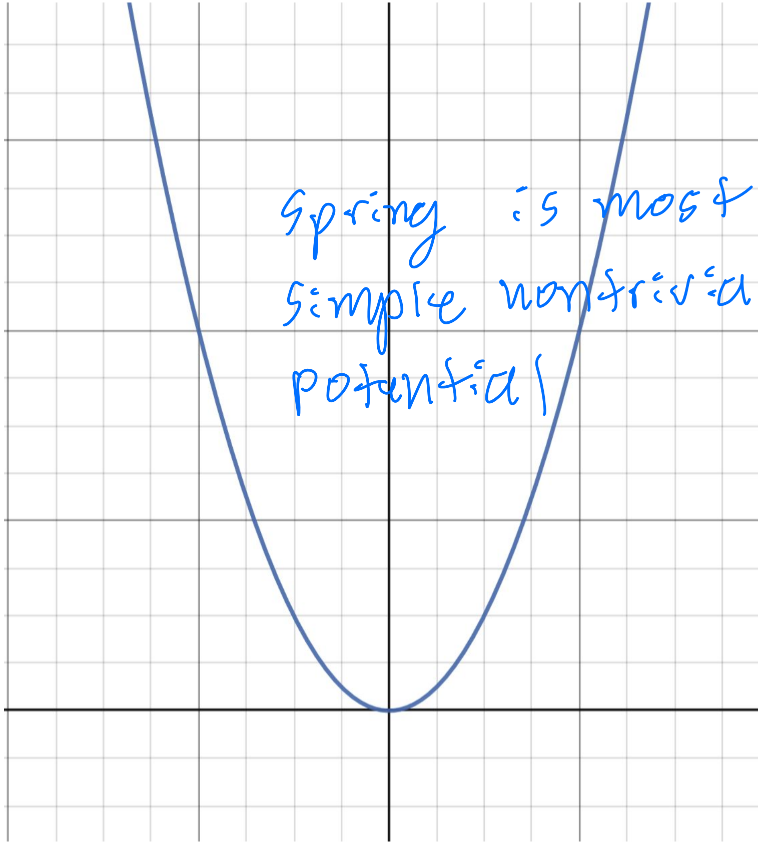
“Energy possessed by an object due to its environment, position, composition, etc.”

$U = U(x)$

Force

$$\boxed{F = -\frac{dU}{dx}}$$

Potential Energy



Spring is most simple nontrivial potential

$$U(x) = \frac{1}{2}kx^2 \longrightarrow F = -kx$$

(Hooke’s Law potential energy) (Hooke’s Law)

Any energy that isn’t attributed to motion (kinetic energy)

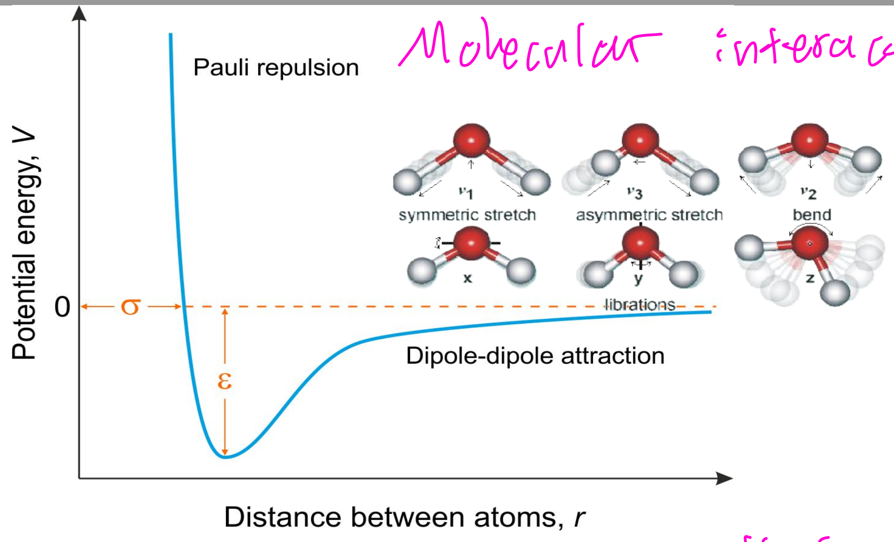
more general eqn of motion:

$$\frac{d^2x}{dt^2} = -\frac{1}{m}\frac{dU}{dx}$$

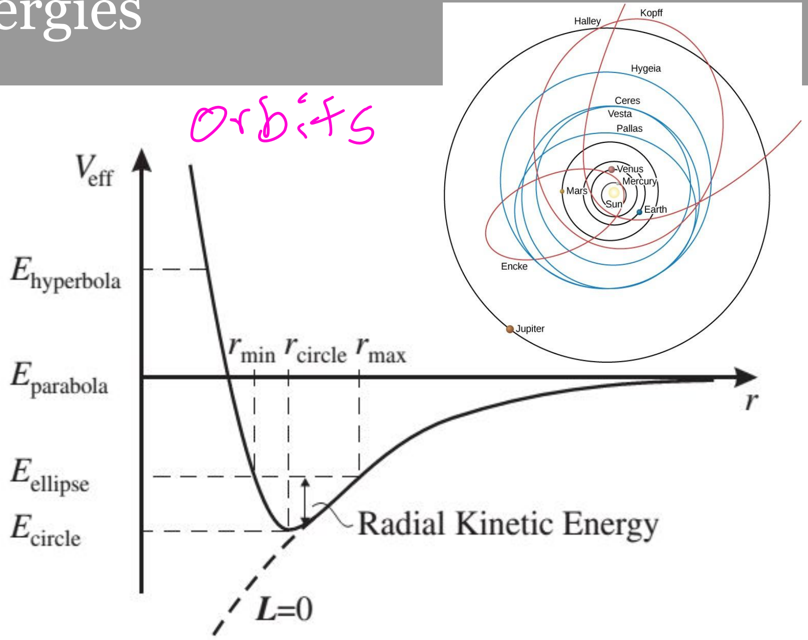
U(x) can be crazy...

Complex potential energies

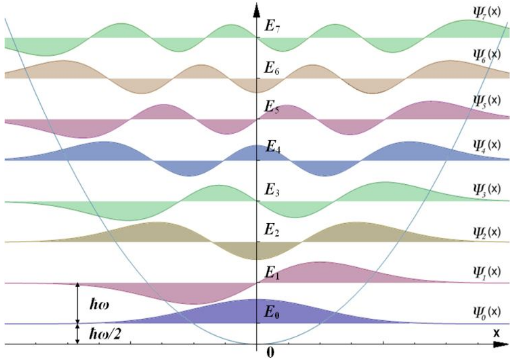
Molecular interactions

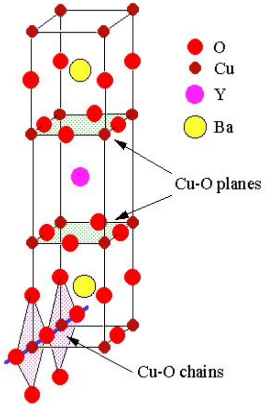


Orbits

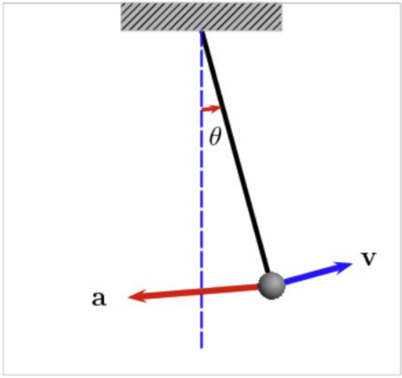


Quantum harmonic oscillators



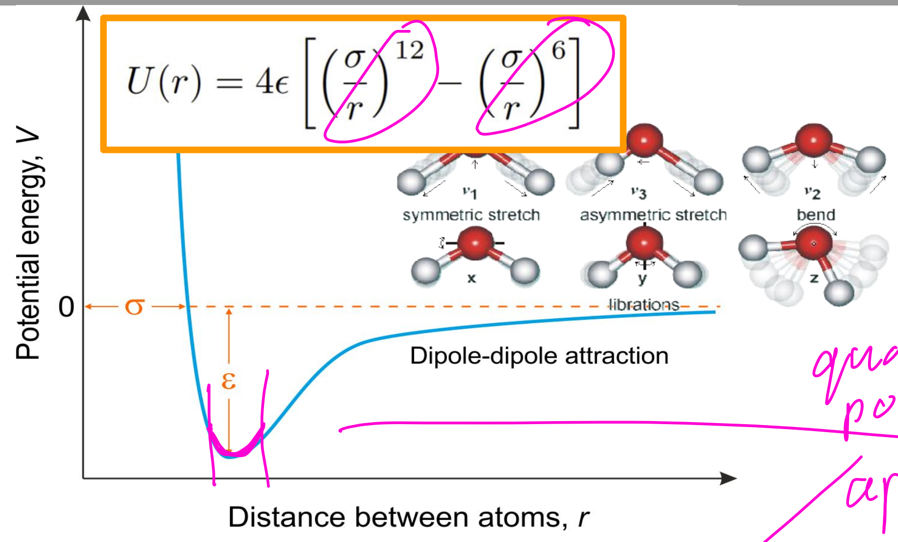


Pendulums

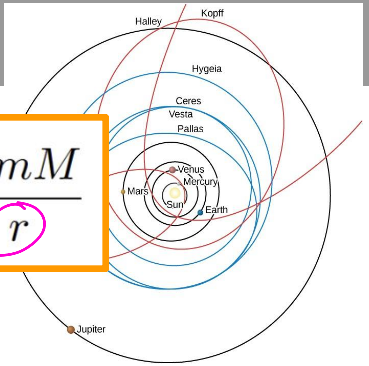


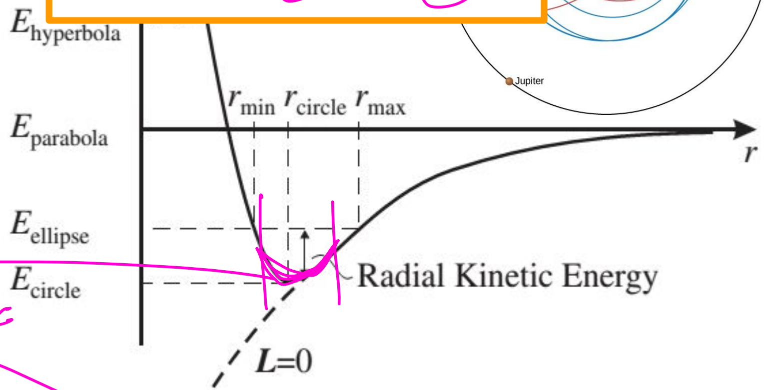
Complex potential energies

$$U(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$



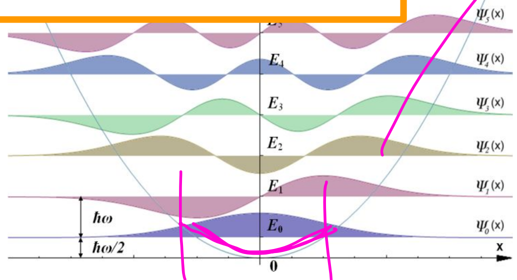
$$U(r) = \frac{L^2}{2mr^2} - \frac{GmM}{r}$$

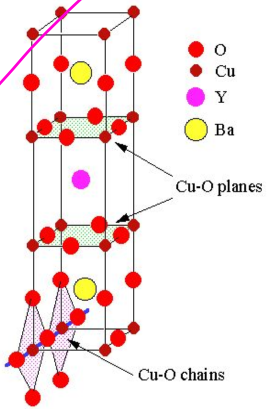




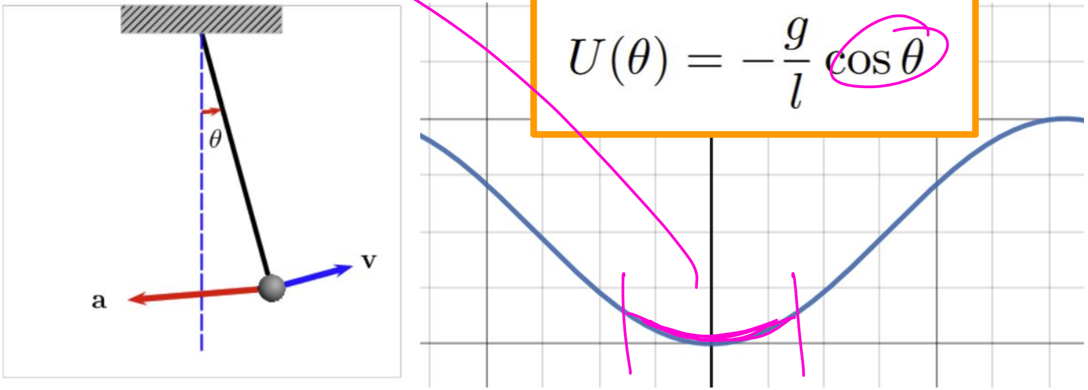
quadratic potential approximations

$$U(x) = \frac{1}{2}\omega^2 x^2$$



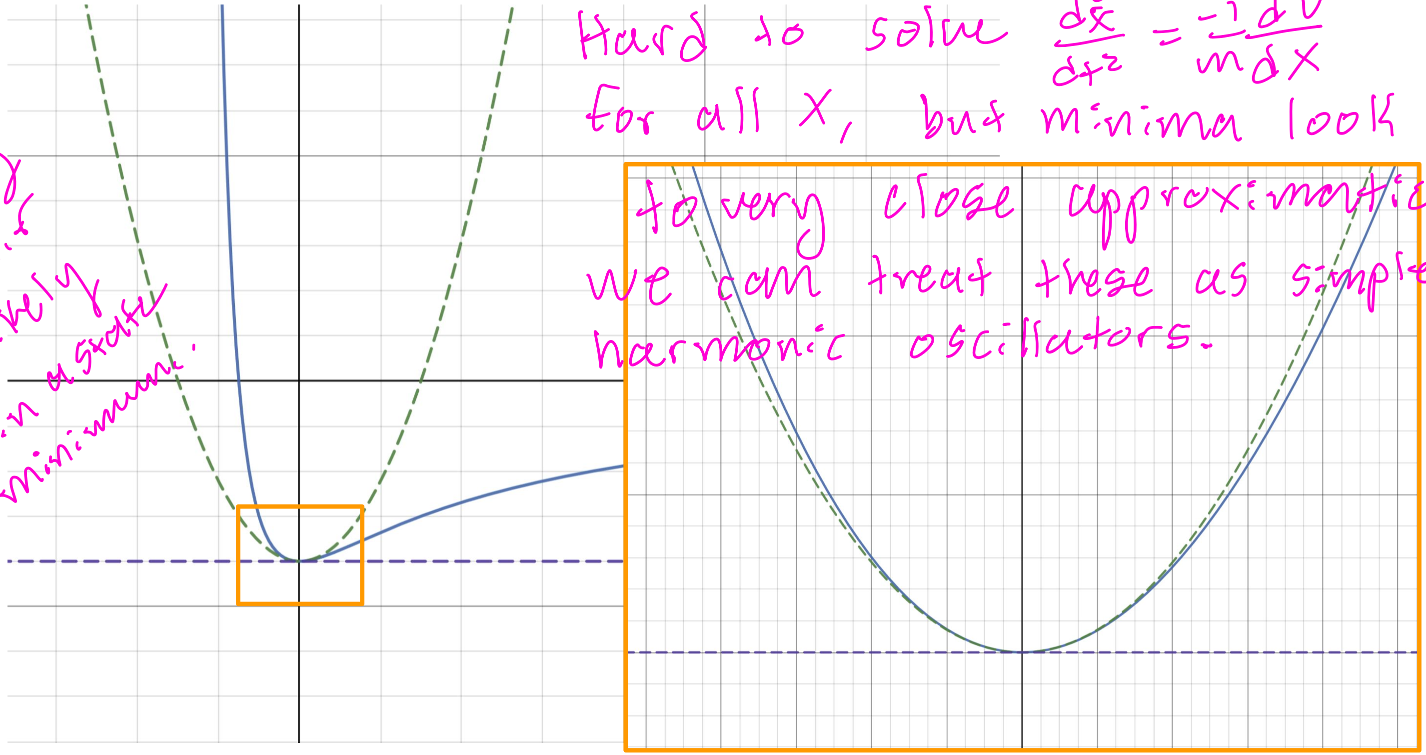


$$U(\theta) = -\frac{g}{l}\cos\theta$$



Taylor expand at the potential minima!!!

$$U(x) = U(x_0) + \frac{U'(x)}{1!}(x - x_0) + \frac{U''(x_0)}{2!}(x - x_0)^2 + \frac{U'''(x_0)}{3!}(x - x_0)^3 + \dots$$



Justified since law of nature that systems try to minimize their energy → very likely a system is in a state close to a minimum.

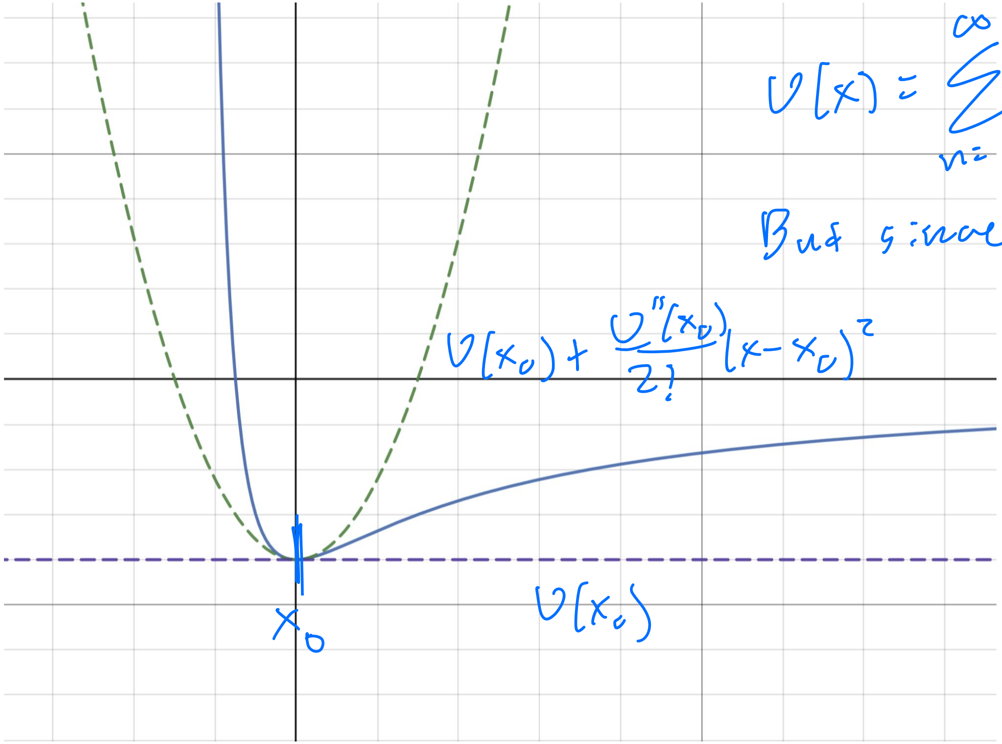
Hard to solve $\frac{d^2x}{dt^2} = \frac{-1}{m}\frac{dV}{dx}$ for all x, but minima look quadratic to very close approximation!

We can treat these as simple harmonic oscillators.

Why everything is a spring

Taylor expand at the potential minima!!!

$$U(x) = U(x_0) + \frac{U'(x)}{1!}(x-x_0) + \frac{U''(x_0)}{2!}(x-x_0)^2 + \frac{U'''(x_0)}{3!}(x-x_0)^3 + \dots$$



$$U(x) = \sum_{n=0}^{\infty} \frac{U^{(n)}(x_0)}{n!}(x-x_0)^n \quad \text{(Taylor series)}$$

But since x_0 @ minima, $U'(x_0) = 0$.

$$\therefore U(x) = U(x_0) + \frac{U''(x_0)}{2}(x-x_0)^2$$

what's the (approximated) eqn of motion then?

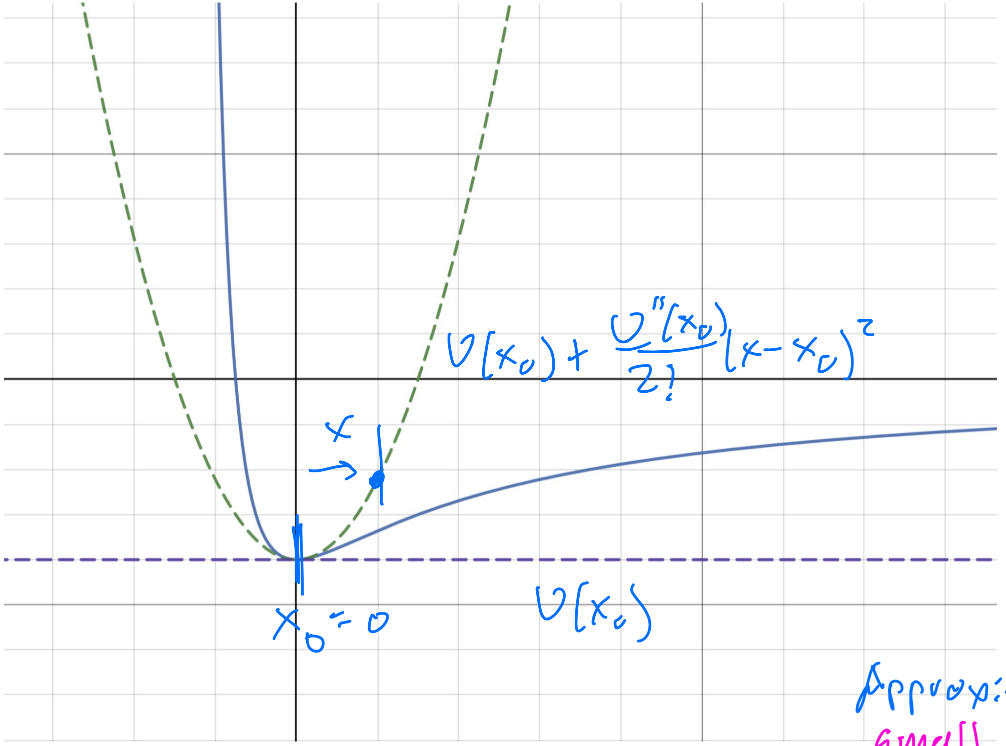
$$F = -\frac{dU}{dt} = -\frac{d}{dt}\Big(\underbrace{\cancel{U(x_0)}}_{\text{const}} + \frac{U''(x_0)}{2}(x-x_0)^2\Big)$$

$$= -U''(x_0)\cdot(x-x_0)$$

Why everything is a spring

Taylor expand at the potential minima!!!

$$U(x) = U(x_0) + \frac{U'(x)}{1!}(x - x_0) + \frac{U''(x_0)}{2!}(x - x_0)^2 + \frac{U'''(x_0)}{3!}(x - x_0)^3 + \dots$$



$x - x_0$ is the displacement from equilibrium! redefine $x - x_0 \rightarrow x$

$$F = -U''(0) \cdot x$$

So what is the eqn of motion?

$$\frac{d^2x}{dt^2} = -\frac{U''(0)}{m} x$$

Simple harmonic motion! Same as Hooke's Law :)

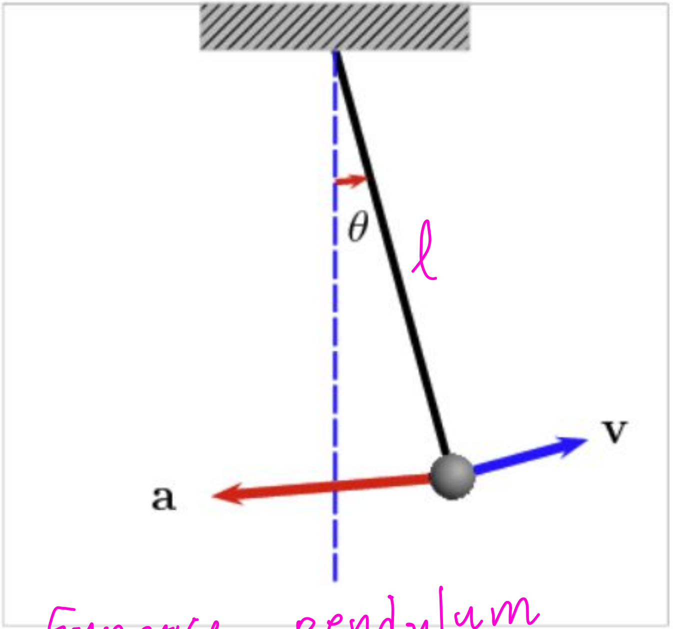
"effective spring constant"

$$K = U''(0)$$

Approximation is only good when $|x - x_0| \ll \frac{U''(x_0)}{U'''(x_0)}$

small displacements + shallow dips

Pendulum



Suppose pendulum starts at an angle θ_0 at $t=0$ with zero initial velocity.

$$\boxed{U(\theta) = -\frac{g}{l}\cos\theta}$$

Taylor / small angle approximation

$$F = -\frac{\partial U}{\partial \theta} = -\frac{g}{\ell}\sin\theta = -\frac{g}{\ell}\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

Solution

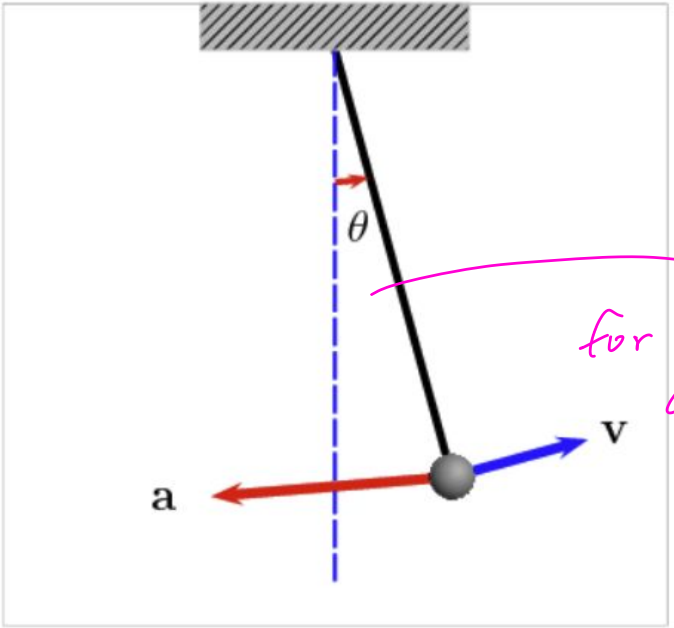
$$\theta(t) = A\cos\left(\sqrt{\frac{g}{\ell}}\,t + \delta\right)$$

$$\theta(0) = \theta_0 = A\cos(\delta) = A \rightarrow A = \theta_0$$

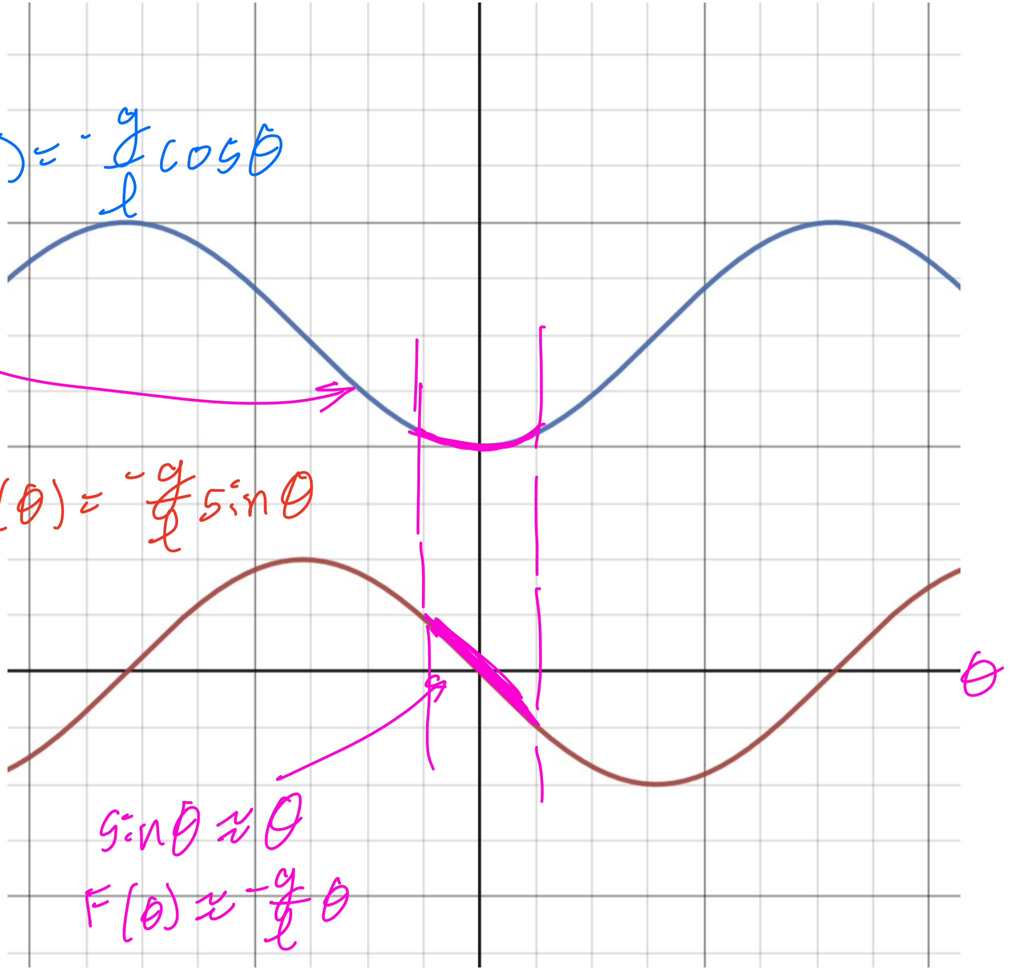
$$\theta'(0) = 0 = -\sqrt{\frac{g}{\ell}}A\sin(\delta) \rightarrow \delta = 0$$

$$\therefore\ \theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{\ell}}\,t\right)$$

Pendulum



for small angles



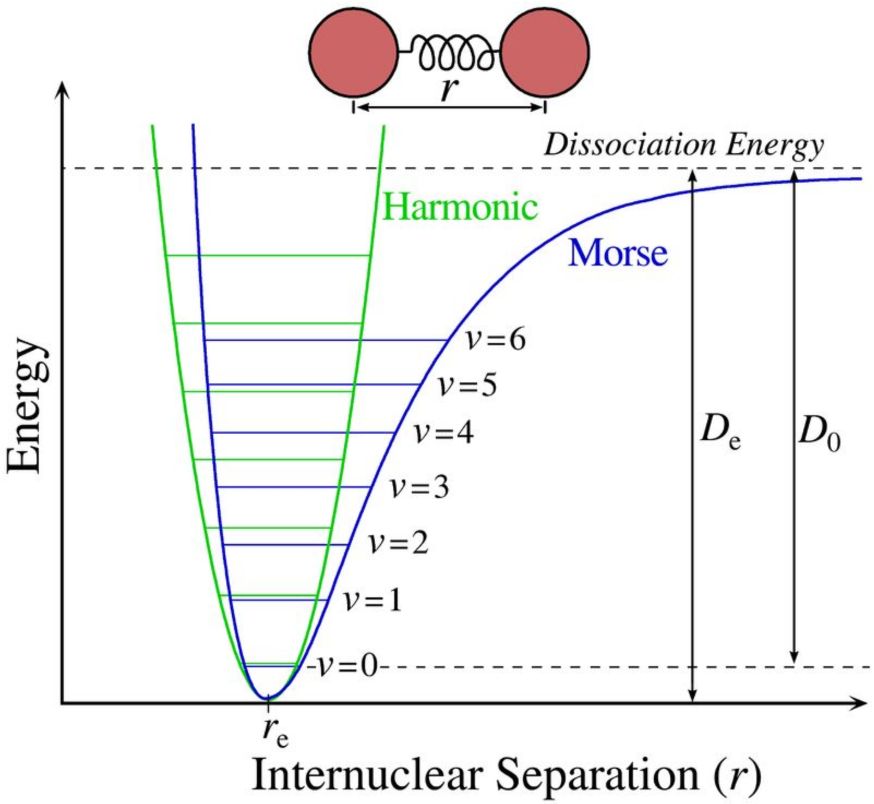
$U(\theta) = -\frac{g}{\ell}\cos\theta$

$F(\theta) = -\frac{g}{\ell}\sin\theta$

$\sin\theta \approx \theta$

$F(\theta) \approx -\frac{g}{\ell}\theta$

Interatomic interactions



"Lennard-Jones potential"

$$U(r) = \frac{\alpha}{r^{12}} - \frac{\beta}{r^6}$$

1. where is the minimum

$$U'(r) = 0 \rightarrow \frac{-12\alpha}{r^{13}} + \frac{6\beta}{r^7} = 0$$

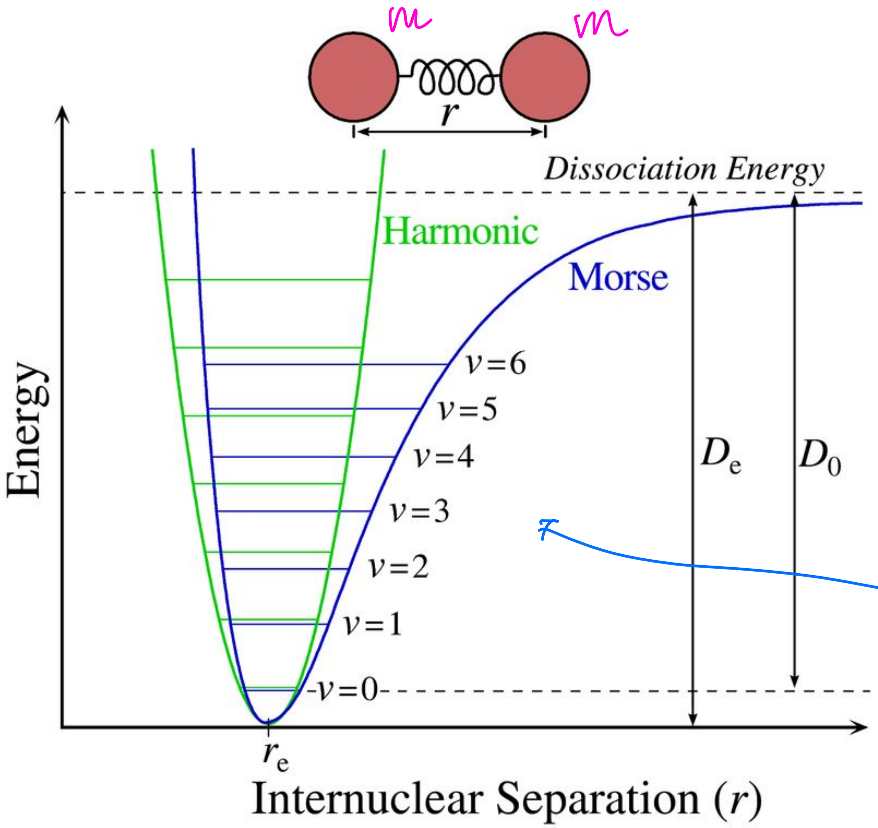
$$r_{min} = \left(\frac{2\alpha}{\beta}\right)^{1/6}$$

2. what is the effective spring constant here?

$$k = U''(r_{min}) = \frac{12\cdot 13\alpha}{r^{14}} - \frac{6\cdot 7\beta}{r^8}\Bigg|_{r=\left(\frac{2\alpha}{\beta}\right)^{1/6}}$$

$$= \frac{9\cdot 2^{2/3}\beta^{7/3}}{A^{7/3}}$$

Interatomic interactions



3. what is the angular frequency?

$$\omega = \sqrt{\frac{k}{m}} = 3\sqrt{\frac{2^{1/3} B^{7/3}}{m A^{7/3}}}$$

$$T = \frac{\omega}{2\pi} = \frac{3}{2\pi}\sqrt{\frac{m A^{7/3}}{2^{1/3} B^{7/3}}}$$

harmonic potential closely resembles actual potential

- Harmonic oscillators show up in many places in physics
- Very accurate approximations of wide variety of systems
- Sill many systems where the approximation falls apart

Thank you!

$a > 0$ (a positive smile)

$a < 0$ (a negative frown)