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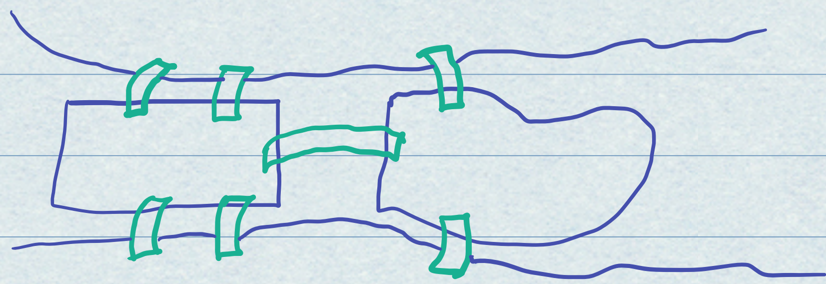
  & undergrad at Berkeley

- A historical problem
  The Pregel river ran through Königsberg, Prussia. In it, there were two large islands connected to each other & the mainland by 7 bridges.

  Let's draw a picture:



The problem: Can we come up with a walk (path) passing each bridge EXACTLY once? TRY THIS!!!
& end up where we started?

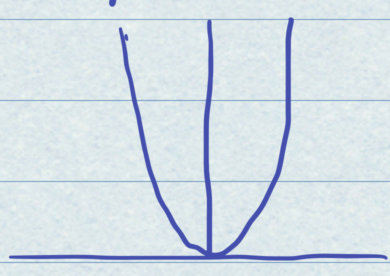
Euler tried to solve this problem, concluding his work in 1736. [old problem]

His main struggle was coming up with an appropriate tool to study the problem with.

Euler's new tool was early Graph Theory

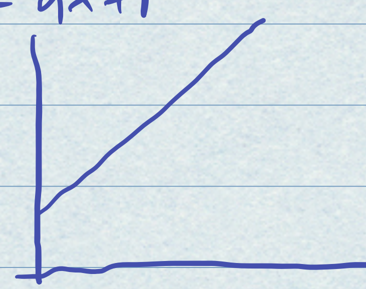
So what is a graph?
You may have graphed equations in school, so when you hear "graph", you think of

$y = x^2$



OR

$y = 2x + 1$



The type of graph we define today is simpler & less rigid.

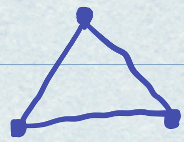
Before we tackle our first problem, I really want you to get familiar with our new tool.

Def: Graph

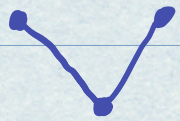
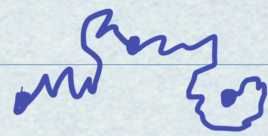
A Graph (Denoted $G(V,E)$) is a collection of vertices (nodes) V, & a set of edges E. Each edge is associated with 1 or 2 vertices, called end points.

This is effectively connect the dots on a grand scale.

But lets look at some examples!

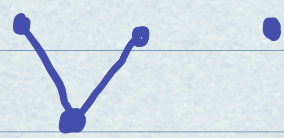
• Ex 1)  is the same as 

Moral: A Graph is just nodes & connections.

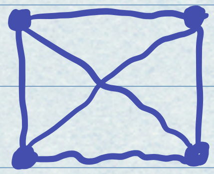
• Ex 2)  is the same as 

& both are acceptable graph:

Moral: Not all vertices need to be connected.

↳ ESCALATION: 

is a valid graph, although it is more fruitful to analyze these two separately.

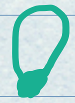
• Ex 3)  is a graph.

Note there is no node where the "middle segments" cross. this is okay.

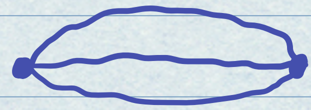
Moral: edges can cross without having a node.

Ex 4) [NON Ex] 
is NOT a graph: edges must have 2 end points!

But how then, as in our definition, can an edge be associated with exactly one node?

Loops!  is a valid graph!

Ex 5) Complicating things:

 is a valid graph!

i.e. two nodes may be connected by multiple edges.

We are now ready to dive in to the world of graphs.

- TRY: Take a minute & try to come up with 3 valid graphs.
  ↳ Think about what sort of "graph" properties you see.
  ↳ come up to the board & draw some!

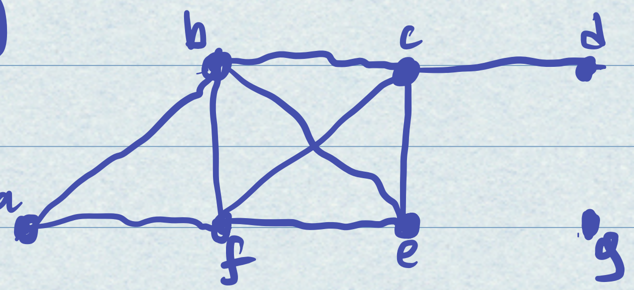
~ Graph Terminology ~

Def: Adjacency:
Two vertices a & b in a graph G(V,E) are adjacent (or neighbors) if a & b are the endpoints of some

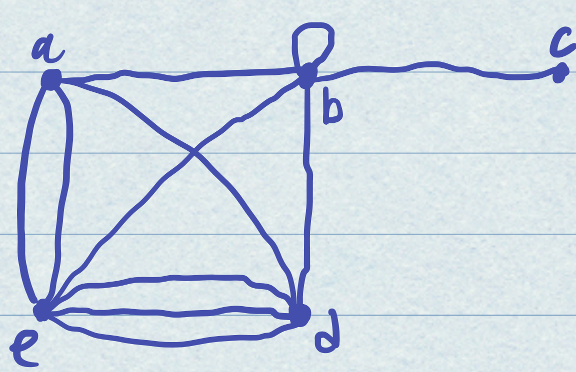
edge in E. We say such an edge is incident with a & b & that it connects a & b.

Def: Degree:
The degree of a vertex is the number of edges incident with it.
WE COUNT LOOPS TWICE.
If v is a vertex, we say that the degree of v is $\deg(v)$.

1)



2)



What is the degree of each vertex here?
$\Rightarrow$ ASK!

It turns out the degree of a graph will be a key fact. Let's study it!

In particular, the sum of the degrees of a graph has special properties.

Let $V = \{\text{vertices of our graph}\}$
Then $\sum_{v \in V} \deg(v)$ is the sum of our degrees.

for 1) it's $2+4+4+1+3+4+0$
$= 18 = 2 \cdot \underline{9}$

for 2) it's $4+6+1+5+6$
$= 22 = 2 \cdot \underline{11}$

Anything special about these 2 numbers? The # of edges! CHECK!
(one of your graphs)

Conjecture: If we have m edges, & V is our set of vertices, then
$2m = \sum_{v \in V} \deg(v)$.

This is TRUE! (Handshaking lemma!)
how can we prove it?

Now we can begin solving graph theoretic problems!

Q: In a graph, how many vertices can have odd degree? [CHECK].

Thm: A Graph has an even number of vertices of odd degree

Let V_1 be the set of vertices with even degree. Let V_2 be the set of vertices with odd degree.

By the hand shaking lemma,

$$2m = \sum_{v \in V} \deg(v)$$

$$= \underbrace{\sum_{v \in V_1} \deg(v)}_{\text{even}} + \underbrace{\sum_{v \in V_2} \deg(v)}_{\text{even}}$$

$\Rightarrow$

But the only way we add odds to get even is if there is an even number of them.

We're almost ready for Königsberg!

The last tool we need is PATHS.

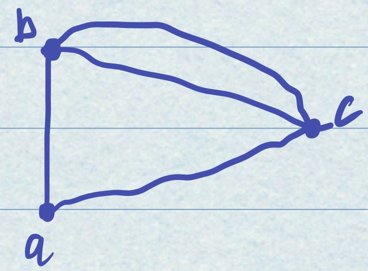
Def: path

Let $n \geq 0$ & let G be a graph. A path of length n from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ where the exist nodes $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of nodes s.t. e_1 has end points x_0, x_1 & in general e_i has end points x_{i-1} & x_i.

We call these two notations EDGE form & VERTEX form resp.

NOTE: When we have multiple edges btw 2 nodes, vertex form is confusing:

Ex)



What is the path abc? TWO WAYS :)
⇒ Use EDGE form!

Def: A path is simple if we don't pass over any edge twice. ⇒ Draw Ex

Def: A CIRCUIT is a path that begins & ends at the same vertex. ⇒ Draw Ex

Def: A simple graph has no loops or multiple edges ⇒ Draw simp & non simp.