Berkeley Math Circle:
a collection of problems & BAMO preparation

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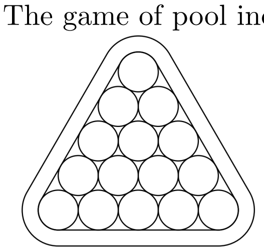
Problem 1.
There are 99 baskets, each containing some apples and some bananas. Prove the following: no matter how many apples and bananas each basket contains, one can always choose 50 baskets, such that these 50 baskets (in total) contain at least half of all the apples and half of all the bananas.

Hint: Consider a simpler problem: there are 3 baskets, and show that one can always choose 2 of them so that they (in total) contain at least half of all the apples and half of all the bananas.

For the next three problems, consider the ***pigeonhole principle****: if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.*

Problem 2. (BAMO 2022)
The game of pool includes 15 balls that fit within a triangular rack as shown:



Seven of the balls are "striped" (not colored with a single color) and eight are "solid" (colored with a single color). Prove that no matter how the 15 balls are arranged in the rack, there must always be a pair of striped balls adjacent to each other.

Problem 3.
Arbitrarily choose $n+1$ different numbers from $1, 2, \ldots, 3n$. Prove that there are always two numbers (among these $n+1$ numbers), whose difference is $\geq n$ and $\leq 2n$.

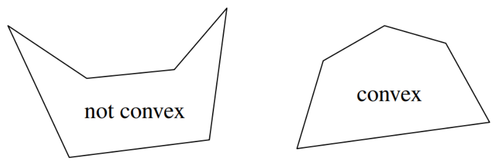
Problem 4. (BAMO 2010)
Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red. Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).

- At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

For the next three problems, you may use ***proof by contradiction****: show that assuming the statement to be false leads to a contradiction.*
For Problem 5 and 6, consider "extremal objects".

Problem 5. (BAMO 2022)
A polygon is called convex if all of its internal angles are smaller than 180°. Here are examples of nonconvex and convex polygons:



Given a convex polygon, prove that one can find three distinct vertices A, P, and Q, where PQ is a side of the polygon, such that the perpendicular from A to the line PQ meets the segment PQ (possibly at P or Q).

Problem 6.
There are n points in the plane, satisfying the following: for any two of them, the straight line passing through them also passes through a third point (that is among these n points). Prove that all these n points are in the same line.

Problem 7. (BAMO 2014)
A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings. It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n-\sqrt{2k}$ and no more than $n+\sqrt{2k}$ games.

Problem 8.
(a) Pick three random points in a circle uniformly. What is the probability that they are contained in a semi-circle?

(b) Pick n random points in a circle uniformly. What is the probability that all these n points are contained in a semi-circle?

(c) Pick n random points in a sphere uniformly. What is the probability that all these n points are contained in a semi-sphere? How about higher dimensions?

Hint: Consider pairs of points that are symmetric by the center of the circle.

Detailed explanation: https://www.mathpages.com/home/kmath327/kmath327.htm

Problem 9. (IMO 1977)
In a finite sequence of real numbers, the sum of any 7 successive terms is negative and the sum of any 11 successive terms is positive. Determine the maximum number of terms in the sequence.