Similarities I

Properties of similarities (discussed in class)

1. A homothety is a similarity.
2. Similarities *preserve lines*, i.e. if f is a similarity and points A, B, C are collinear, then the points $f(A), f(B)$, and $f(C)$ are also collinear.
3. Similarities *preserve circles*.
4. Similarities *preserve angles* between lines, and therefore send similar polygons to similar polygons.
5. Any similarity is the composition of an isometry and a homothety with the same factor, i.e. if f is a similarity with the factor k, then there is an isometry g such that $f = H_C^k \circ g$.

First applications (discussed in class)

1. Given a square on the plane and an arbitrary point X. Show that the points X_1, X_2, X_3, and X_4, symmetric to X with respect to the midpoints of the sides of the square, also form a square.
2. Construct a square inscribed into a given semicircle.
3. Given two concentric circles, construct a line on which the circles cut out three congruent segments.

Some more applications (to practice on your own)

1. Given two lines and a point A, construct a circle tangent to the lines and passing through A.
2. In a given triangle, construct two circles of equal radii such that each circle is tangent to the other one and to two sides of the triangle.
3. On the sides AB and BC of $\triangle ABC$, find points X and Y such that $|BX| = |XY| = |YC|$.

New properties of homotheties (to be discussed in class)

1. Let S and S' be two circles with radii r and r'. Show that if $r \neq r'$, then there are *exactly two* homotheties, one with $k > 0$ and one with $k < 0$, that transform S into S'.
   What changes when $r \neq r'$?
2. Let ABC and $A'B'C'$ be two triangles with parallel sides, $AB||A'B', BC||B'C'$, and $CA||C'A'$. Show that if $|AB| \neq |A'B'|$, then there is *a unique* homothety H_O^k that takes $\triangle ABC$ to $\triangle A'B'C'$.
   What transformation can be used to take $\triangle ABC$ to $\triangle A'B'C'$ when $|AB| = |A'B'|$?
3. Let ℓ be a line. Denote by ℓ' the line obtained from ℓ under the homothety H_O^k.
   (a) Show that the new line ℓ' is either parallel to ℓ or coincides with it.
   (b) Describe all lines preserved by H_O^k (i.e. those ℓ for which ℓ' coincides with ℓ).
4. Let f be a transformation that sends each line ℓ to a line ℓ' which is either parallel or equal to ℓ.
   (a) Show that if f has no *fixed points* (i.e. $f(A) \neq A$ for all A), then f *must be a translation*.
   (b) Show that if f a fixed point A, then f is *must be a homothety* with the center A.
5. Let H_A^k and H_B^ℓ be two homotheties and let $f = H_B^\ell \circ H_A^k$ be *their composition*.
   (a) Show that if $k\ell \neq 1$, then f is a homothety $H_C^{k\ell}$ whose center C belongs to the line AB.
   (b) Show that if $k\ell = 1$, then f is a translation in the direction parallel to the line AB.