BMC-Advanced **February 8, 2023**

Quadratic irrationalities II

1. Find the last digit of the number $\lfloor(\sqrt{2}+\sqrt{3})^{100}\rfloor$.
   (Here $\lfloor x \rfloor$ denotes the integer part (the floor) of number x)t.

2. Find the largest power of 2 which divides $\lfloor(1+\sqrt{3})^{99}\rfloor$.

3. Show that $(1+\sqrt{2})^{2023}$ can be written as $\sqrt{N}+\sqrt{N+1}$ for some integer N.

4. Show that the following equations have infinitely many integer solutions:
   (a) $x^2 = 2y^2 - 1$; (c) $x^2 - 3y^2 = 4$; (c) $x^2 - 17y^2 = 1$.
   (d) What about the equation $x^2 - 3y^2 = 2$?

5. Show that *every* integer solution of the equation $|x^2 - 2y^2| = 1$ is given by $x + y\sqrt{2} = (1+\sqrt{2})^k$ for some integer k.
   [Hint. Verify first that if $|x^2 - 2y^2| = 1$, then $|y| \le |x|$ and $|x| < 2|y|$.]

6. Find continued fractions expression for (a) $\sqrt{3}$; (b) $\sqrt{10}$; (c) $\sqrt{17}$.
   I.e. find positive integers $a_0, a_1, a_2, \ldots$ such that $\sqrt{3} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$, etc.

7. Try to find a connection between the previous problem and solutions of the equations $x^2 - dy^2 = 1$ for $d = 3,$ 10, and 17.