Berkeley Math Circle **230913** **Graph Theory I**

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Definitions

1 A *graph* is a pair (V,E), where V is a finite set and E is a set of unordered pairs of elements of V. The elements of V and E are called *vertices* and *edges*, respectively. A graph is usually understood to be *simple*, having no multiple edges and no loops. Otherwise, it is called a *pseudograph*.

2 A *directed graph (digraph)* is the same as a graph, except that the edges are now ordered pairs of distinct vertices. (An edge is said to "come out of" the first vertex in the pair and "go into" the second vertex.) When we say "graph", we mean an undirected graph unless otherwise specified.

3 An *isomorphism* of two (directed or undirected) graphs is a bijection of their vertices such that two vertices of one graph form an edge if and only if their images form an edge.

4 Two vertices are *adjacent* if they are the endpoints of an edge. The *degree* of a vertex is the number of edges it is an endpoint of. In a directed graph, the *in-degree* and *out-degree* of a vertex are the number of edges coming in and going out of, respectively, that vertex. A graph is *k-regular* if every vertex has degree k.

5 A *walk* is a sequence vertex, edge, vertex, ..., which ends with a vertex, and where the edge between any two vertices in the sequence is an edge which actually joins those two vertices. In other words, a walk is just what you think it is. The *length* of a walk is the number of edges in the walk. If the starting vertex is the same as the ending vertex, the walk is *closed*.

6 A walk with no repeated edges is called a *trail*. A walk with no repeated vertices is called a *path*.

7 A closed trail is called a *circuit*. A "closed path" is a contradiction in terms, but what this term evokes is called a *cycle*. More precisely, a cycle is a closed walk in which no vertex is repeated except for the starting vertex (which is the same as the end vertex). A cycle of length n is called an *n-cycle*.

8 A graph is *connected* if for any two vertices, there exists a walk starting at one of the vertices and ending at the other. Otherwise the graph is called *disconnected*.

9 A connected graph with no cycles is called a *tree*. If the graph is disconnected, and each connected component is a tree, then the entire graph is called a *forest*.

10 If the vertices of a graph can be partitioned into two (non-empty) subsets such that all edges of the graph connect only vertices from different sets (never two vertices from the same sets), then the graph is called *bipartite*.

11 The *complete* graph K_n is the graph on n vertices in which every pair of vertices is an edge. The *complete bipartite* graph $K_{m,n}$ is the graph on $m+n$ vertices in which every pair of vertices, one from the first m and one from the other n, is an edge.

12 A *planar* graph is one that can be drawn in the plane, with points representing the vertices, and (polygonal) curves representing the edges, so that no two edges meet except at a common endpoint. The regions into which the edges divide the plane are called *faces*.

13 An *Eulerian trail/circuit* is a trail/circuit which visits every edge of a graph. Such a graph is called Eulerian.

14 A *Hamiltonian path/cycle* is a path/cycle which visits each vertex of the graph. Such a graph is called Hamiltonian.

Several Basic Theorems

1 *Handshake Lemmma*. The sum of the degrees of the vertices equals twice the number of edges; as a corollary, if v is odd, one of the vertices has even degree.

2 For connected graphs, $e \geq v-1$, with equality holding for trees. For a forest with k connected components, $e = v-k$.

3 If $e \geq v$, then the graph has a cycle.

4 A graph is bipartite if and only if it has no odd cycles.

5 (a) A graph has an Eulerian trail if and only if it has either zero or two vertices with odd degree.

(b) A graph has an Eulerian circuit if and only if all vertices have even degree.

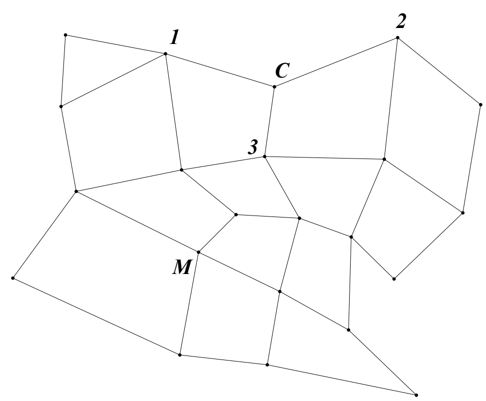
6 For a connected planar graph, $v-e+r=2$, where r denotes the number of regions (including the unbounded region) that the graph divides the plane into.

Three Games that use Graphs

For games 1 and 3, two players alternate turns. The rule for a legal move is described. The game ends when no legal moves can be made. The winner is the last player to make a legal move. Your job is to analyze the game and figure out a winning strategy.

1 *Color the Grids*. You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

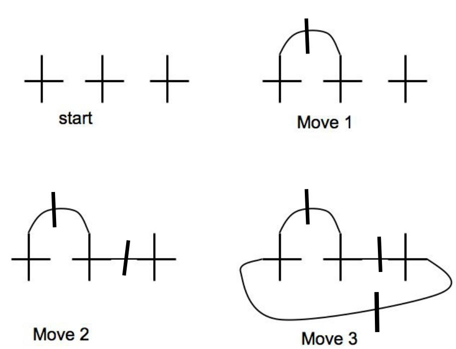
2 *Cat and Mouse*. A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.

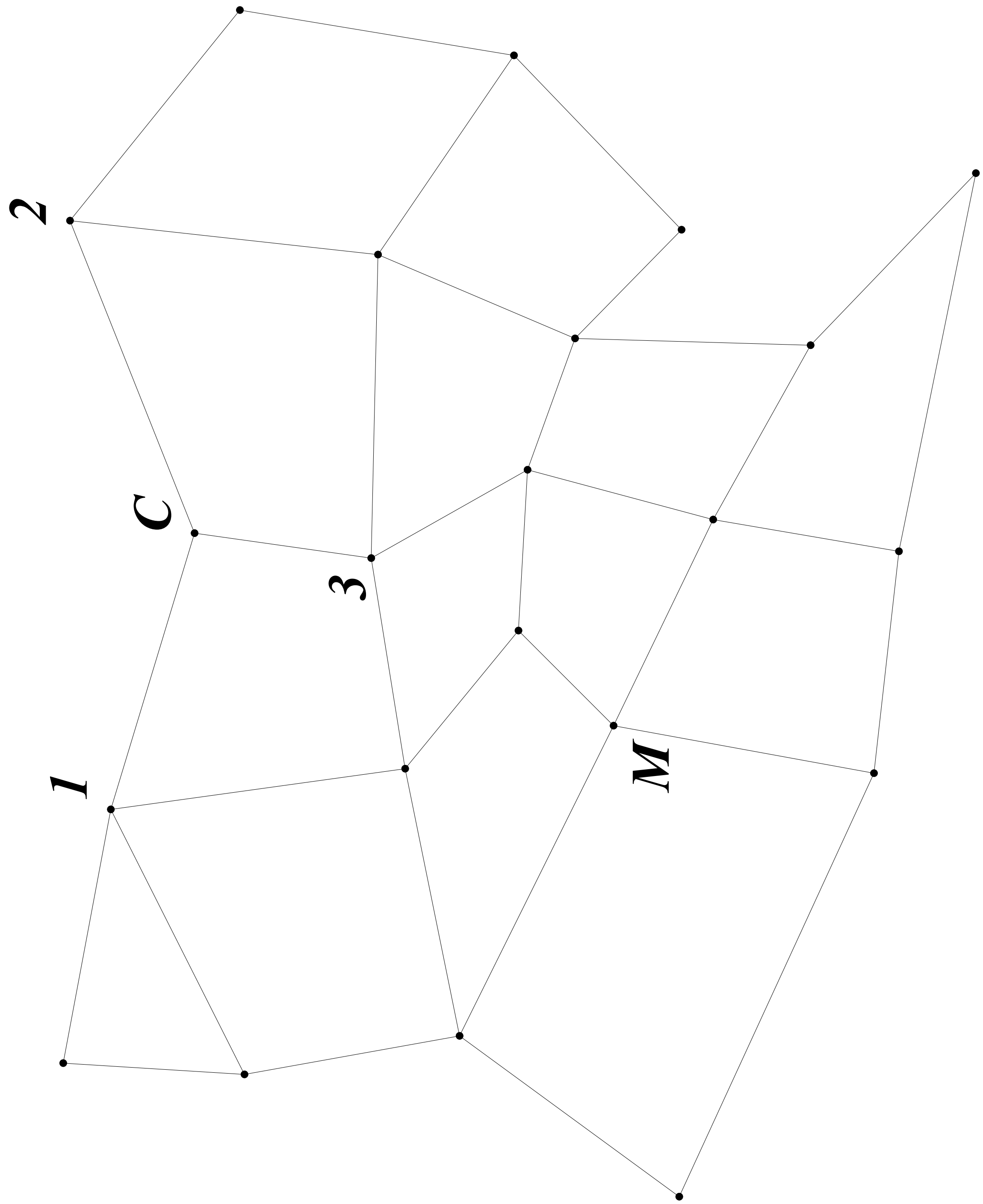


Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. You may move backwards; for example, if the mouse moved to point 1 on its first turn, it could move back to M on the next turn.

The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win? (Adapted from Ravi Vakil's *A Mathematical Mosaic*. A larger diagram is on the next page, so you can play this game.)

3 *Brussels Sprouts*. Start by putting a few crosses on a piece of paper. On each move, a player can connect the two endpoints of a cross together, with a single line (which can be curved). Then a new cross is drawn on this connection line. You cannot ever draw a line that intersects another already-drawn line. Here is an example of the first few moves of a 3-cross game.




2
C
3
1
M

More Problems

4 Show that every graph contains two vertices of equal degree.

5 In the nation of Klopstockia, each province shares a border with exactly three other provinces. Can Klopstockia have 17 provinces?

6 Draw a graph with eight vertices, four of which have degree 4 and four of which have degree 3.

7 Show that it is possible to have a 4-regular graph with n vertices, for every $n \geq 5$.

8 (Colorado Springs Mathematical Olympiad) If 127 people play in a singles tennis tournament, prove that at the end of the tournament, the number of people who have played an odd number of games is even.

9 How many edges must a graph with n vertices have in order to guarantee that it is connected?

10 A large house contains a television set in each room that has an odd number of doors. There is only one entrance to this house. Show that it is always possible to enter this house and get to a room with a television set.

11 A group of people play a round-robin chess tournament, which means that everyone plays a game with everyone else exactly once (chess is a one-on-one game, not a team sport). There are no draws.

(a) Prove that it is always possible to line up the players in such a way that the first player beat the second, who beat the third, etc. down to the last player. Hence it is always possible to declare not only a winner, but a meaningful ranking of all the players.

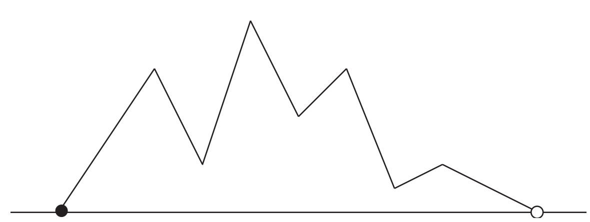
(b) Give a graph theoretic statement of the above.

(c) Must this ranking be unique?

12 Call a rectangle *good* if at least one side has integer length. Suppose a large rectangle is tiled by good rectangles (i.e., perfectly covered by these good rectangles, with no missing area or overlaps). Prove that the large rectangle must also be good.

13 Two people are located at opposite ends of a mountain range, at the same elevation. If the mountain range never drops below this starting elevation, is it possible for the two people to walk along the mountain range and reach each other's starting place, while always staying at the same elevation?

Here is an example of a "mountain range." Without loss of generality, it is "piecewise linear;" i.e., composed of straight line pieces. The starting positions of the two people are indicated by two dots.



14 (USAMO 1986) During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three of them were sleeping simultaneously.