BMC-Advanced: Fourier Analysis

Aaron Agulnick

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1 The Heat Equation

Consider a metal rod bent into a circle, with circumference 2π meters. Call the point at which the ends of the rod are glued together 0, so that we can identify the rod with the unit interval $[-\pi, \pi)$ glued together at the endpoints. Suppose the rod starts with a temperature at each point x given by $f(x)$. We would like to understand the temperature at point x after time t, say as a function $u(t, x)$.

If you were a physicist in the early 1800s, it would have been known to you that such a function describing the heat of the rod must satisfy

$$\begin{cases} \partial_t u = \partial_x^2 u \\ u(0, x) = f(x) \end{cases}$$

at least when you choose your units appropriately. You would also know that when such a u exists, it is unique, so finding any solution to this differential equation actually tells you how the heat of the rod evolves over time.

A reasonable hope is that the function u splits as

$$u(t, x) = v(t)g(x)$$

for some single-variable equations v and g. In such a situation, one could observe

$$\partial_t u = v'(t)g(x) = v(t)g''(x) = \partial_x^2 u$$

or, in another form,

$$\frac{v'(t)}{v(t)} = \frac{g''(x)}{g(x)}.$$

Since one side is a function only of t, and the other is a function only of x, one concludes that both sides must be constant. That is, there exists a constant λ such that

$$\begin{cases} v'(t) = \lambda v(t) \\ g''(x) = \lambda g(x) \end{cases}$$

This method, called **separation of variables**, reduces our partial differential equation to an ordinary differential equation.

These equations should suggest to you some sort of exponential; in particular, $g''(x) = \lambda g(x)$ suggests using $g(x) = e^{inx}$, since this is 2π-periodic when n is an integer, just like our initial state $f(x)$, and it satisfies $g''(x) = -n^2 g(x)$. (It is a theorem that λ must be negative for any reasonable region whose heat evolution we want to understand.)

Once we have obtained $g(x) = e^{inx}$, the separation of variables approach suggests that we should pick $v_n(t)$ with $v_n'(t) = \lambda v_n(t)$, which we can solve for $v_n(t) = c_n e^{-n^2 t}$ for some constant c_n. But c_n has some influence on the initial condition; namely $u(0, x) = c_n e^{inx}$. We have thus obtained

$$u(t, x) = c_n e^{-n^2 t} e^{inx} \tag{1}$$

which satisfies

$$\begin{cases} \partial_t u = \partial_x^2 u \\ u(0, x) = c_n e^{inx} \end{cases} \tag{2}$$

What is the physical interpretation of this?

Rewrite $\partial_t u = \partial_x^2 u \iff (\partial_t - \partial_x^2) u = 0$. Since differentiation is linear, so is $\partial_t - \partial_x^2$. That is, if u and $\tilde{u}$ satisfy $(\partial_t - \partial_x^2) u = (\partial_t - \partial_x^2) \tilde{u} = 0$, then

$$(\partial_t - \partial_x^2)(a_1 u + a_2 \tilde{u}) = a_1 (\partial_t - \partial_x^2) u + a_2 (\partial_t - \partial_x^2) \tilde{u} = 0. \tag{3}$$

Since we already found that $u(t, x) = e^{-n^2 t} e^{inx}$ solves $(\partial_t - \partial_x^2) u = 0$, we have

$$\sum c_n e^{-n^2 t} e^{inx} \tag{4}$$

as a solution to the heat equation with initial condition

$$u(0, x) = \sum c_n e^{inx} \tag{5}$$

for any (finite) sum over integers n. It is not hard to argue that this extends to infinite sums when they converge. Thus, if we have

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \tag{6}$$

then we also have

$$u(t, x) = \sum_{n \in \mathbb{Z}} c_n e^{-n^2 t} e^{inx} \tag{7}$$

satisfying the heat equation with initial condition $u(0, x) = f(x)$. When can f be decomposed in this way?

2 Orthonormality

Suppose for a second that we know

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{8}$$

for appropriate coefficients c_n, called its *Fourier coefficients*. How could we find the c_n?

The key observation is the following:

Theorem 1 (Orthonormality). *$\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = 0$ if $m \neq n$ and 2π if $m = n$.*

Proof. The proof is easy: just compute!

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \int_{-\pi}^{\pi} e^{i(n-m)x}dx. \tag{9}$$

If $n = m$, the integrand is 1, and we have

$$\int_{-\pi}^{\pi} 1\, dx = 2\pi. \tag{10}$$

Otherwise, $n - m = k$ is some other integer, and

$$\int_{-\pi}^{\pi} e^{ikx}dx = \frac{1}{k}[e^{ikx}]_{x=-\pi}^{x=\pi} = \frac{1}{k}(e^{i\pi k} - e^{-i\pi k}) = 0 \tag{11}$$

since e^{ikx} is 2π-periodic. □

Remark 1. The meaning is that the complex exponentials are orthonormal with respect to a certain inner product, which is just

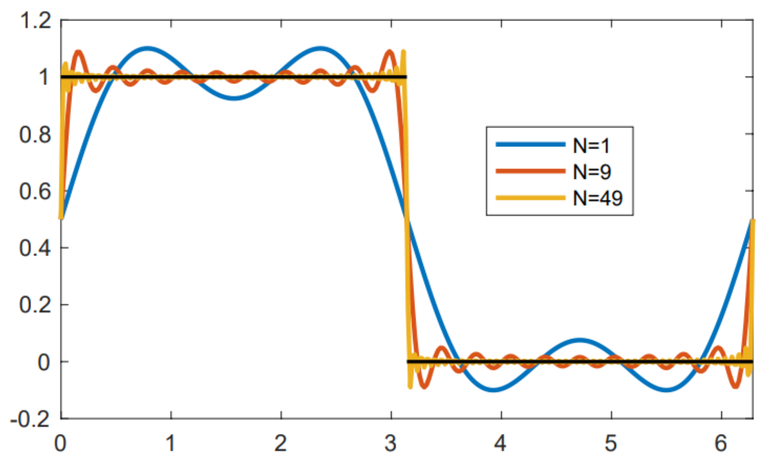
$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx. \tag{12}$$

This is a manifestation of a much more general phenomenon, appearing under the broad umbrella of Pontryagin duality, which unites the discrete Fourier transform, Fourier series, and the Fourier transform on $\mathbb{R}$. The above argument works with only the knowledge of the addition formula $e^{inx}e^{imx} = e^{i(n+m)x}$ and 2π-periodicity.

In any case, we can now extract the Fourier coefficients of functions if we are sufficiently confident that they exist.

Exercise 1. Compute the Fourier coefficients of the step function $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

One obtains a sequence of functions $f_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$ which converge (in some sense) to f as $N \to \infty$. We can graph f_N for increasing values of N and appreciate that they get closer and closer to the desired function. However, one observes *Gibbs' phenomenon*: that at the points of discontinuity of f, the sequence of functions seems to be a much worse approximation than elsewhere. The image below is not precisely the function described above, but it is illustrative.



(Obtained from these lecture notes.)