1 The Heat Equation

Consider a metal rod bent into a circle, with circumference $2\pi$ meters. Call the point at which the ends of the rod are glued together 0, so that we can identify the rod with the unit interval $[-\pi, \pi)$ glued together at the endpoints. Suppose the rod starts with a temperature at each point $x$ given by $f(x)$. We would like to understand the temperature at point $x$ after time $t$, say as a function $u(t, x)$.

If you were a physicist in the early 1800s, it would have been known to you that such a function describing the heat of the rod must satisfy

$$\begin{cases}
\partial_t u = \partial_x^2 u \\
u(0, x) = f(x)
\end{cases}$$

at least when you choose your units appropriately. You would also know that when such a $u$ exists, it is unique, so finding any solution to this differential equation actually tells you how the heat of the rod evolves over time.

A reasonable hope is that the function $u$ splits as

$$u(t, x) = v(t)g(x)$$

for some single-variable equations $v$ and $g$. In such a situation, one could observe

$$\partial_t u = v'(t)g(x) = v(t)g''(x) = \partial_x^2 u$$

or, in another form,

$$\frac{v'(t)}{v(t)} = \frac{g''(x)}{g(x)}$$

Since one side is a function only of $t$, and the other is a function only of $x$, one concludes that both sides must be constant. That is, there exists a constant $\lambda$ such that

$$\begin{cases}
v'(t) = \lambda v(t) \\
g''(x) = \lambda g(x)
\end{cases}$$
This method, called separation of variables, reduces our partial differential equation to an ordinary differential equation.

These equations should suggest to you some sort of exponential; in particular, \( g''(x) = \lambda g(x) \) suggests using \( g(x) = e^{inx} \), since this is \( 2\pi \)-periodic when \( n \) is an integer, just like our initial state \( f(x) \), and it satisfies \( g''(x) = -n^2 g(x) \). (It is a theorem that \( \lambda \) must be negative for any reasonable region whose heat evolution we want to understand.)

Once we have obtained \( g(x) = e^{inx} \), the separation of variables approach suggests that we should pick \( v_n(t) \) with \( v'_n(t) = \lambda v_n(t) \), which we can solve for \( v_n(t) = c_n e^{-n^2 t} \) for some constant \( c_n \). But \( c_n \) has some influence on the initial condition; namely \( u(0, x) = c_n e^{inx} \). We have thus obtained

\[
\dot{u}(t, x) = c_n e^{-n^2 t} e^{inx}
\]

which satisfies

\[
\begin{cases}
\partial_t u = \partial_x^2 u \\
u(0, x) = c_n e^{inx}
\end{cases}
\]

What is the physical interpretation of this?

Rewrite \( \partial_t u = \partial_x^2 u \iff (\partial_t - \partial_x^2) u = 0 \). Since differentiation is linear, so is \( \partial_t - \partial_x^2 \). That is, if \( u \) and \( \tilde{u} \) satisfy \( (\partial_t - \partial_x^2) u = (\partial_t - \partial_x^2) \tilde{u} = 0 \), then

\[
(\partial_t - \partial_x^2)(a_1 u + a_2 \tilde{u}) = a_1 (\partial_t - \partial_x^2) u + a_2 (\partial_t - \partial_x^2) \tilde{u} = 0.
\]

Since we already found that \( u(t, x) = e^{-n^2 t} e^{inx} \) solves \( (\partial_t - \partial_x^2) u = 0 \), we have

\[
\sum c_n e^{-n^2 t} e^{inx}
\]

as a solution to the heat equation with initial condition

\[
u(0, x) = \sum c_n e^{inx}
\]

for any (finite) sum over integers \( n \). It is not hard to argue that this extends to infinite sums when they converge. Thus, if we have

\[
f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx},
\]

then we also have

\[
u(t, x) = \sum_{n \in \mathbb{Z}} c_n e^{-n^2 t} e^{inx}
\]

satisfying the heat equation with initial condition \( u(0, x) = f(x) \). When can \( f \) be decomposed in this way?
2 Orthonormality

Suppose for a second that we know

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \]  

for appropriate coefficients \( c_n \), called its Fourier coefficients. How could we find the \( c_n \)?

The key observation is the following:

**Theorem 1 (Orthonormality).** \( \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = 0 \) if \( m \neq n \) and \( 2\pi \) if \( m = n \).

**Proof.** The proof is easy: just compute!

\[ \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \int_{-\pi}^{\pi} e^{i(n-m)x} dx. \]

If \( n = m \), the integrand is 1, and we have

\[ \int_{-\pi}^{\pi} 1 dx = 2\pi. \]

Otherwise, \( n - m = k \) is some other integer, and

\[ \int_{-\pi}^{\pi} e^{ikx} dx = \frac{1}{k} [e^{ikx}]_{x=-\pi}^{\pi} = \frac{1}{k} (e^{i\pi k} - e^{-i\pi k}) = 0 \]

since \( e^{ikx} \) is \( 2\pi \)-periodic. \( \square \)

**Remark 1.** The meaning is that the complex exponentials are orthonormal with respect to a certain inner product, which is just

\[ (f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx. \]

This is a manifestation of a much more general phenomenon, appearing under the broad umbrella of Pontryagin duality, which unites the discrete Fourier transform, Fourier series, and the Fourier transform on \( \mathbb{R} \). The above argument works with only the knowledge of the addition formula \( e^{inx} e^{imx} = e^{i(n+m)x} \) and \( 2\pi \)-periodicity.

In any case, we can now extract the Fourier coefficients of functions if we are sufficiently confident that they exist.

**Exercise 1.** Compute the Fourier coefficients of the step function \( f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \)
One obtains a sequence of functions $f_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$ which converge (in some sense) to $f$ as $N \to \infty$. We can graph $f_N$ for increasing values of $N$ and appreciate that they get closer and closer to the desired function. However, one observes Gibbs' phenomenon: that at the points of discontinuity of $f$, the sequence of functions seems to be a much worse approximation than elsewhere. The image below is not precisely the function described above, but it is illustrative.

(Obtained from these lecture notes)