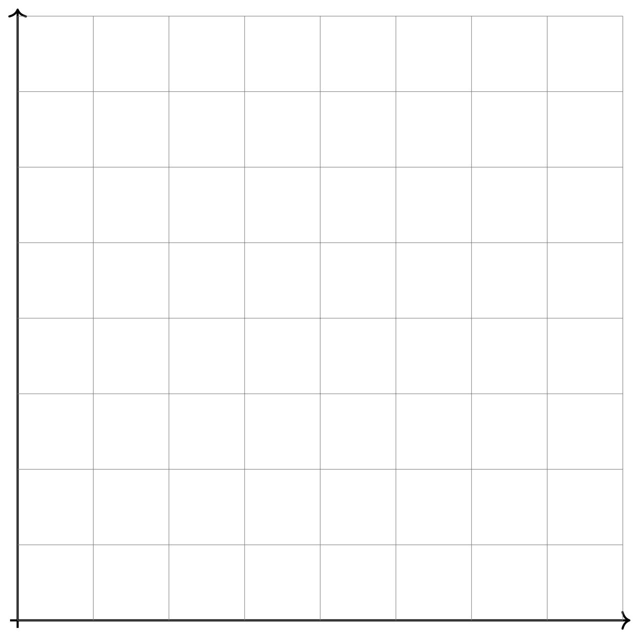
Hat Problems II

BMC Adv Fall 2022

December 7, 2022

1 Cardinalities

1. The cardinality of a finite set A is the number of elements inside A and is denoted $|A|$. What is $|\{a, b, c, d, e, \ldots, z\}|$?
2. We say that two sets A and B have the same cardinality or $|A| = |B|$ if there is a bijection from one to the other. Give an example of a bijection between $\{1, 2, 3, 4, 5\}$ and $\{a, b, c, d, e\}$ and another function between them that is not a bijection.
3. Prove that the cardinality of the even numbers is equal to the cardinality of the odd numbers.
4. We say that a set is countably infinite if $|A| = |\mathbb{N}|$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Prove that the even numbers are countable.
5. Prove that the integers are countable.
6. Prove that if A, B are both countable, then their union $A \sqcup B$ is also countable.
7. Divide $\mathbb{N}$ into four sets A, B, C, D that are each infinite and disjoint and so that $A \cup B \cup C \cup D = \mathbb{N}$.
8. Divide $\mathbb{N}$ into a countably infinite number of sets $A_1, \ldots, A_n, \ldots$ that are each infinite and disjoint so that $\bigcup_i A_i = \mathbb{N}$.



9. Prove that the squares in the first quadrant are countable.
10. Prove that the rational numbers $\mathbb{Q}$ are countably infinite.
11. A racecar on an infinite track starts at some integer location and has some constant integer speed per second. Every second, you can check to see if the race car is at one location on this infinite track. What is a strategy that guarantees that you will eventually find the car?
12. The set $2^{\mathbb{N}}$ is the set of all infinite sequences of 0s and 1s, so one element would be $(0, 1, 0, 0, 1, 1, 0, \dots)$. Prove that $2^{\mathbb{N}}$ is uncountable using Cantor's diagonal argument.
13. Prove that the real numbers $\mathbb{R}$ are uncountable.
14. Prove that we can divide a circle up into a countably infinite set of identical pieces.

2 Hat Problems

1. Suppose that two people have an infinite number of black and white hats stacked on their heads. They must then guess a position on their head and if both positions have black hats, then they each get a million dollars. What is their probability of winning if they both guess randomly? What if they use the strategy of picking the location of the first black hat on the other person's head?
2. There are a countably infinite number of ministers lined up with white and black hats on so they can see everyone in front of them. At the same time, they guess what color hat they have. What strategy can they use so that only a finite number of them are incorrect?
3. Starting from the back, they are now to guess what color hat they have in order and everyone can hear the guesses of everyone behind them. Is there a strategy so that at most one of them is incorrect?
4. If one person has an infinite number of black and white hats stacked on their head and has to guess a position on their head that is a black hat. What is a strategy so that he wins 100% of the time?