Advanced Hat Problems I

Wednesday, November 30, 2022 5:19 PM

1. Three people w/ either a black or white hat on.
→ First, someone asks, "Who can see a black hat", and everyone raises their hand

Then someone asks, "Who knows their own hat color?" No one raises their hand, then one person raises their hand. What color is their hat, and how did they know?

Different scenarios:

~~W W W~~	~~B W W~~	B W B	B B B
1	2	3	4

We know that there are ≥ 2 black hats, in Scenario 3, if you have a black hat, you see 1 W, 1 B. So 2 people will raise their hands.

Since no one raises their hands ⇒ In scenario 4 so everyone has black hat.

2. ? ? ? →

2B 1W or 2W 1B

~~B W W~~
~~W B B~~

opposite

? B W
? W B

knows

3. ? ? ? ? ? ? ? ? ? ? →

If everyone guesses randomly, $P = 1/2^{10}$

If paired up $P = 1/2^5$

Strategy: 1^{st} person says W if odd # white hats
B if even # white hats

W = odd = 1
B = even = 0

? B B W W W B W W B

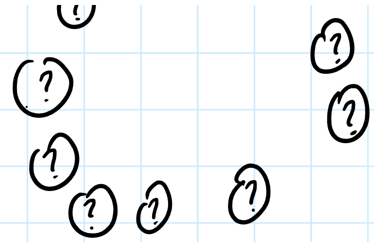
Say:	W	B	B	W	W	W	B	W	W	B
P(correct)	50%	100%	100%	100%	-	-	-	-	-	100%

4. ? ? ? ?

Want ≥ 5 right

Strategy 1 Say majority color seen.

If b Black and w White, b > w then



Strategy 1 Say majority color seen.
If b Black and w White, $b > w$, then
people will see $b-1$ B, w W   Everyone says B
b B, $w-1$ W $\Rightarrow b > 5$ right

Problem: 5W 5B
Have black hat 5W 4B Say W
white hat 5B 4W Say B } everyone wrong.

Probability success $= 1 - \frac{\binom{10}{5}}{2^{10}} = 1 - \frac{252}{1024} \approx \frac{3}{4}$

Strategy 2: Divide group into 2, half guess total # white hats = even
half guess total # white hats = odd.
One of the two halves is right $\Rightarrow$ 100%, 5 people are right.

Strategy 3: Pair them up. One person says the other person's hat color
Other person says opposite the hat color.

	B	W	B	B
Say	W	W	B	W

5. King says "I see 10 white hats"
Split into two groups w/ equal # white hats

Strategy Split into 10 and $n-10$ people

	10		$n-10$
White	w	$10-w$	$10-w$
Black	$10-w$	w	$n+w-20$

↑ Reverse hats

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3 people w/ hats, all say at the same time {B, W, Pass}
If no one guesses incorrectly and ≥ 1 person guess correct, then they win.

$a = 00001$
$b = 00010$
$c = 00011$
$d = 00100$

binary: $10110 = 1\cdot 16 + 0\cdot 8 + 1\cdot 4 + 1\cdot 2 + 0\cdot 1$
$= 22$
16 8 4 2's 1's place

g = 00011
d = 00100

16 8 4 2's 1's place

hello = binary?
8 5 12 12 15

h e l l o
01000**1** 00101**0** 01100**0** 01100**0** 01111**0**
010101
j

Q How can we identify that an error has occurred?

A Add a parity bit, if there are an odd # 1's add a 1
even # 1's add a 0

What kind of errors can a parity bit detect? Do we know what the original message is?
Odd # bits changed — No.

010101

110101	000101	011101	010001	010111	010100
z	b	m	h	k	j

Mistake could occur in any 5 original bits, or the parity bit

Q If we want to send 1 bit so that if one bit is changed, you will still be able to figure out the original bit, how can I do this?

If I send 000 → receive 000, 100, 010, 001
111 → receive 111, 011, 101, 110

<u>Definition</u> The Hamming distance between 2 bitstrings is the # of bits that are different.

Ham(3,1) code = (000, 111)

<u>Definition</u> A Hamming code is a set of bitstrings so that every bitstring is ≤ 1 Hamming distance from a Hamming code.

Ham(7,4) uses 7 bits to encode 4 bits.

$p_1 p_2 d_1 p_3 d_2 d_3 d_4$ $p_1 = d_1 + d_2 + d_4$ $p_3 = d_2 + d_3 + d_4$
$p_2 = d_1 + d_3 + d_4$

Message = 1010 $p_1 = 1+0+0 = 1$ $p_2 = 1+1+0 = 0$ $p_3 = 0+1+0 = 1$

Encoded = 1 0 1 1 0 1 0

Encode hello 1000, 0101, 1100, 1100, 1111 ⇒ 1110000, 0100101
0111100, 1111111

Decode (w/ error) 1011011, 1010011, 1110111

Way to decode, look new message bits = 1011, compute parity bits = $\overset{p_1'}{0}\ \overset{p_2'}{1}\ \overset{p_3'}{0}$

The incorrect bit is at position $\underset{p_3 - p_3'}{1}\ \underset{p_2 - p_2'}{1}\ \underset{p_1 - p_1'}{1}{}_2 = 7$ ⇒ original message 1011010

1010011 $\xrightarrow{\text{new message}}$ 1011 $\xrightarrow{\text{new parity bits}}$ 010 $\xrightarrow[\text{reverse order}]{\text{compare in}}$ 011 = 3 → original message 1000011.

Every wrong code is 1 distance from real message → Information Theory

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Back to hats

Strategy 1: 2 people pass, one person guesses → 50% success

Strategy 2: See 2 different colored → pass
2 same colored → opposite color

Hats	Guess	
BBB	WWW	X
BBW	PPW	✓
BWB	PWP	✓
BWW	BPP	✓

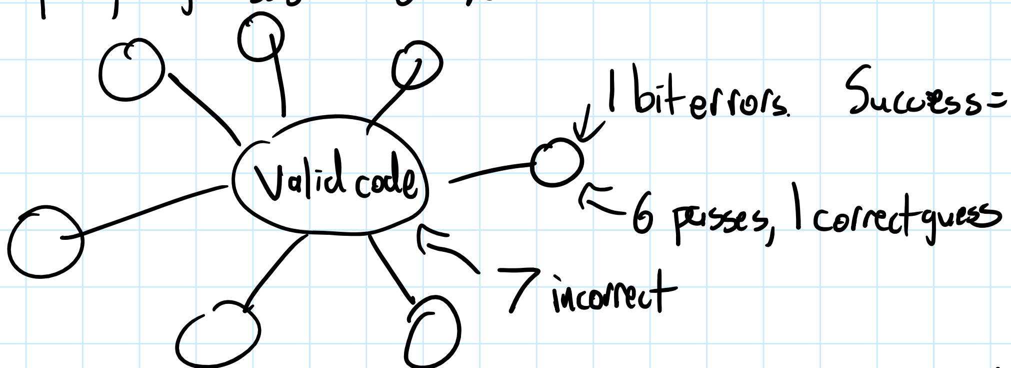
3/4 success.

Similar to Ham(3,1). Strategy: Assign people positions. If your hat could make a valid code, guess the opposite.

What if 7 people play this game?

Strategy 1: 6 pass, 1 guesses → 50% success.

Strategy 2:



Success = $7/8$

In general, there is a hamming code for 2^n-1 bits. → success $= \frac{2^n-1}{2^n}$.

Optimal strategy found + proven for $n = 2, 3, 4, 5, 6, 7, 8, 2^n-1$.