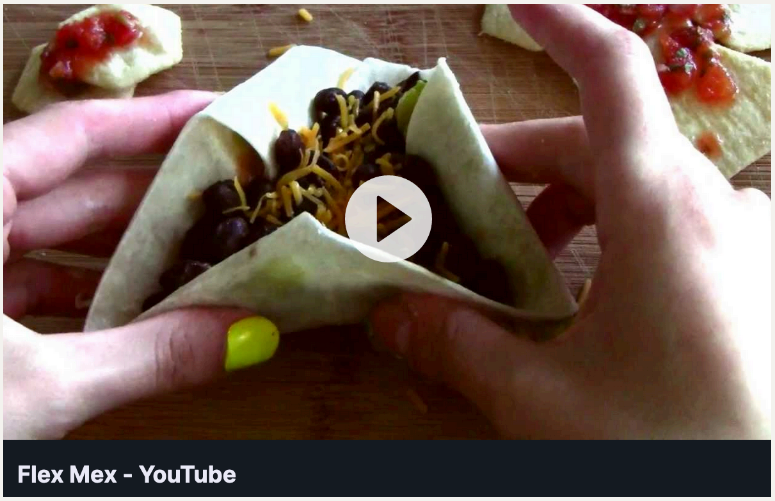
Berkeley Math Circle - Intermediate 2: Flexagons

230428 - Chris Overton (handout after lecture)

When you see: <---- SPOILER ALERT---->, or <-----> , please challenge yourself to answer before reading on.

Plan for today:

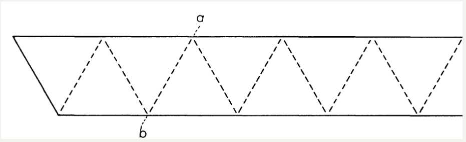
- The fun of discovery: build 3-sided and 6-sided hexaflexagons
- How can we describe these objects "mathematically"?
- What generalizations are interesting?
- **Literature review**: variety of papers, from 'recreational' (e.g. Gardner) to 'serious', and how that varied over time



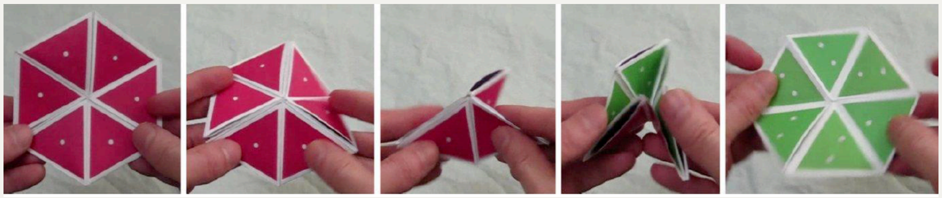
https://www.youtube.com/watch?v=GTwrVAbV56o
This "flex mex" video is hilarious, but part of a very informative series

Setup

We started by building 3 and 9-sided "hexaflexagons". If you can't follow these instructions, try the references below - especially from Martin Gardner



- 3-sided hexaflexagon (my notation: 6#3 for six tiles, 3 "sides")
    - Make a strip of 9 equilateral triangles as above, and mark the direction of the strip on the first and last
    - After every group of 3, fold over
      (consistently! - say counterclockwise turns)
    - Tape the ends (previously marked!) together in such a way that the resulting object has rotational symmetry (per 120 degrees.)
- 6-sided hexaflexagon (6#6)
    - Make a strip of 18 equliateral triangles & mark ends
    - "Roll over" in consistent direction, forming a strip of 9 triangles
      (Fold over nonoverlapping adjacent pairs of triangles, resulting in a straight strip of half the original length.)
    - Follow instructions for 3-sided above
- In both cases, label sides (Hint: mark each triangle's orientation to center!)
    - To expose new sides, use the "pinch flex" operation illustrated below (only works for some folds)
    - HOW DO YOU DESCRIBE THIS THING MATHEMATICALLY?
    - HOW DOES THIS GENERALIZE? (You may construct additional models if you have time)



Setup hints

- Direction of folding matters. You want 3-way rotational symmetry!
  For the 6#3 above, as seen from one side, all three "pouches" should open either in the same direction.
- Some instructions show extra "tab" pieces. Here, we do not glue pieces over each other, we just tape their ends. This focuses on how they are attached. Make sure you connect to form a continuous stip in the two examples above

Engineering

- Pre-fold (probably both ways) to ensure continuing flexibility
- Careful to avoid rips or accumulated misfoldings!
- How to make equilateral triangles? (BMC Elementary...)
- Clever tricks: scaffolds around which to wrap triangles...

6#3 ("tri-hexaflexagon")

- What is your first attempt at a "mathematical" description?
- What does this leave out?
- "Leaves" (aka "panels") vs "stacks" (aka "pat"s)
- What information in a description is redundant, because of how this is assembled?
- After you have labeled all sides, what do you learn by cutting the tape and unrolling? Are you sure you can reconstruct & re-tape as you had it originally?

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You should discover three sides, which are obtainable by successive "pinch flexes" as 1 --> 2 --> 3 --> 1, etc.

6#6 ("hexa-hexaflexagon")

- How many sides should you expect to find?
- Are they equally easy to find? Why?
- How can you capture the "structure" of your creation?
    - Critique this description!
- How do you know whether you have built the same structure as others int the class?
- After you have labeled all sides, can you cut tape, un-roll, study, and re-connect as you had it before?

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You should expect to find 6 sides, because 18 triangles * two sides of paper / 6 sides per hexagon.

In fact, you should be able to find them all if you folded correctly, but interestingly, they're not all equally easy to get to.

6#3, 6#6, and beyond...

- How do structures of 6#3 and 6#6 fit together?
- For which n do you now know how to make 6#n (i.e. n-sided hexaflexagons)?
- Which of these are 'interesting?'
- Assuming you can make any $6\#(3 * 2^n)$, can you make $6#n for n any non-negative integer?

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- We saw that 6#3 EMBEDS in 6#6, and that there are three ways to do this.

- If it is possible to make $6\#(3 * 2^n)$, it is easy to get fewer sides, just by gluing sides together for one face - but careful that you do this only when these are "innermost" sides, so as not to make other sides inaccessible.
- And by the way you can $6\#(3 * 2^n)$, by just repeating the "rolling" operation as at the beginning of 6#6 instructions.
- Thus it is possible to make $6\#k$ for $k \geq 2$, because a flat regular hexagon is just a $6\#2$.

Hexaflexagon history, starting 1939

- Arthur Stone, grad student at Princeton
- The "Flexagon Committe" joined by:
    - Bryant Tuckerman, math graduate student
    - Richard P. Feynman, physics graduate student
    - John W. Tukey, early math instructor
- These worked out original theory of 6#3 and 6#6...

Hexaflexagon "literature" and "theory"

- Recreational literature, such as by Martin Gardner. His first trial paper resulted in (and became the first contribution to) his famous column in Scientific American, compiled in "Hexaflexagons and other Mathematical Diversions", (1959, 1988) available at: https://www.maa.org/sites/default/files/pdf/pubs/focus/Gardner_Hexaflexagons12_1956.pdf

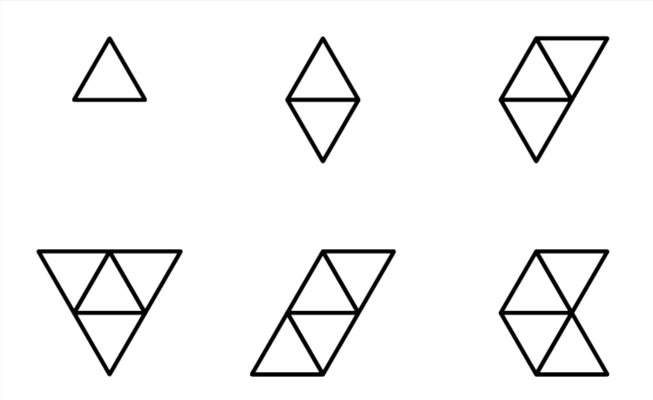
- First 'serious' write-up in 1957 by Oakley & Wisner:
  http://geofhagopian.net/mfd/flexagon01.pdf
  Much denser, a bit counterintuitive, harder to follow, but precise
- Friendlier write-up by Susan Goldstine, who had covered this in an informal grad student seminar at Harvard:
  https://www.gathering4gardner.org/g4g12gift/Goldstine_Susan-Hexaflexagons.pdf

  ... (half a century passes)
- Combinatorics article in 2008 by Anderson et al:
  Rigorous, more modern exposition, with theorems:
  https://www.sciencedirect.com/science/article/pii/S0195669809000699
- There are many "recipes" with printable tamplates at:
  https://flexagon.net/

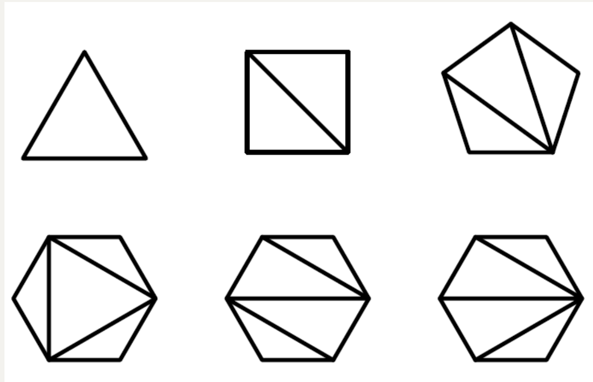
Hexaflexagon theory

What is the right diagram? (The next few images are taken from Goldstine)

1] "cleaned up" triangles



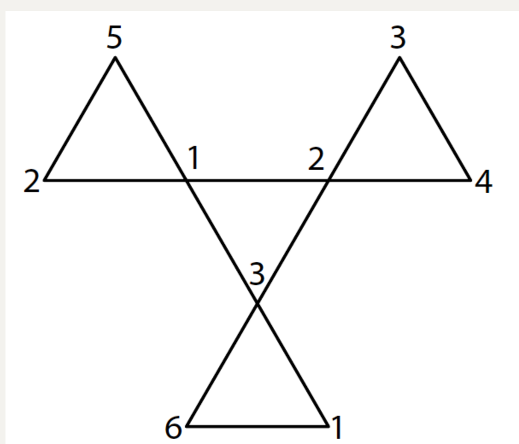
2] Feynman's graphs



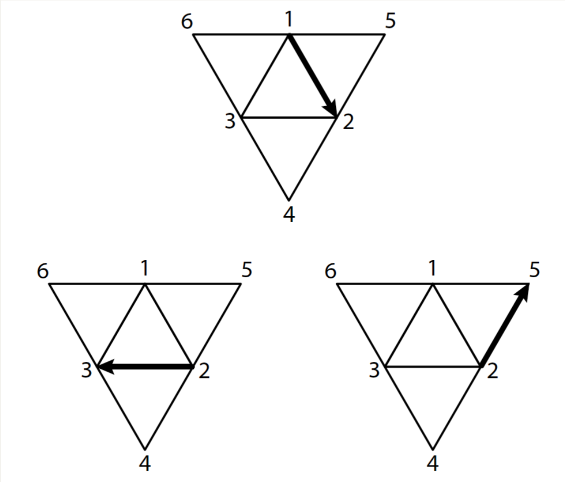
3] Tuckerman's graphs

This shows that with 1 on top and 3 on bottom, the side(s) you can get to next (either a 1 or else a choices of 2 or 5) depends somehow on the configuration of the sides, which is worked out in a complicated way by Oakley & Wisner.

If each triangle is called a "leaf", a hexagon consists of six "stacks" of these (which they call "pats".) The entire geometry of the flexagon is described by how leaves connect to each other in single stacks, and between adjacent stacks.



4] Goldstine's graphs



What are the rules that the graphs try to convey?

It turns out graphs (made just of points and disjoint line segments connecting them) are NOT the best description.

For example, in the graph above, if side 2 is on top and 1 is at bottom, where can you go next? It turns out you can get to 5 or to 3, but not to 4.

Some very common mathematical technology helps here: you need a 2-dimensional **simplicial complex**, namely generated by 2-simplexes (triangles), 1-simplexes (their edges), and 0-simplexes (points: their corners.) These are all **oriented**, which for 2-simpliexes could be shown e.g. by a counterclockwise arc shown in the triangle.

If you add these to each triangle in a Goldstine graph (#4 above), you have a clearer mathematical description. (This does leave out orientation of the triangles in each hexagon, but we'll ignore that for now.)

Here is what the original "flexagon committe" worked out in the 1950's.
A "terminal position" is one that only allows one way forward via pinch flex - not two possibilities as you often see with 6#6.

The essential observations about how the "Feynman" map for any hexaflexagon reflects its structure are as follows.

1. A hexaflexagon with *n* faces has exactly *n* terminal positions, not counting which of the faces is on the bottom or the top. (If you count the order, there are *2n* terminal positions.)

2. Each of the *n* terminal positions corresponds to one of the *n* panels in the hexaflexagon. In particular, up to the order of the faces, the 6 terminal positions in the hexaflexagon in Figures 4 and 5 are (1,6), (2,4), (3,4), (1,5), (2,5), and (3,6). Thus, we can read the panels off of the exterior edges of the diagram.

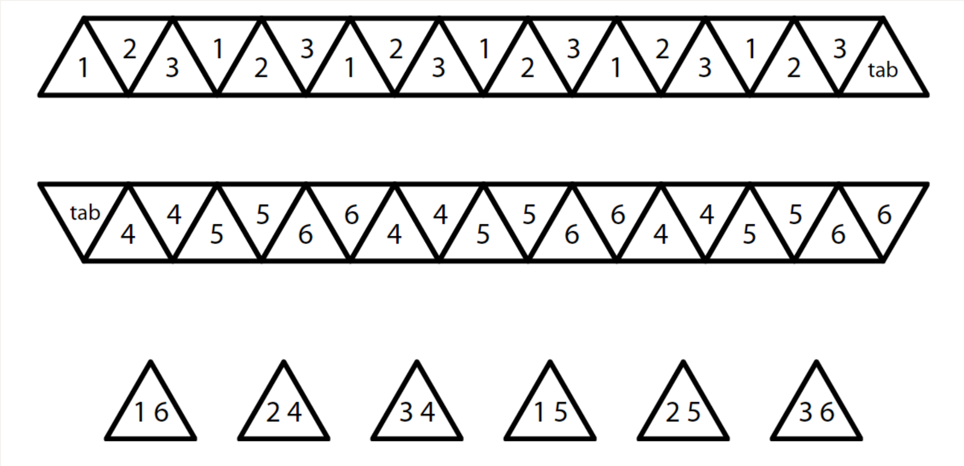
3. In each position, the sets of exterior edges on either side of the edge marking that position correspond to the panels in each of the two distinct pats. For instance, when the hexaflexagon in Figures 4 and 5 is in the position (1,2), one pat contains panels (1 5) and (2 5), and the other contains panels (1 6), (3 6), (3 4), and (2 4). When we flex to position (2,5), the panel (1 5) moves from the thinner pat to the thicker one.

In particular:

- The next visible face after a flex can only be the third side of a triangle containing the current "edge" (top and bottom sides.)
    - There are either 1 or two ways to flex from a current top side
- One thing this leaves out: orientation of the triangles on one side (three possibilities)
- How generally can you prove these patterns? (see papers!)

Just for reference, here is a paper labeling to construct the "standard" hexahexaflexagon 6#6, in a numbering that fits with the graphs we have shown above.

Instead of using an extra "tab" triangle, notice that the left triangle becomes a 1-6 leaf after gluing. If we tape instead, we can then omit the leaf at right.



Note: if you glue together three pairs of adjacent side 6's, nothing stops you from folding the other sides into a 6#5 - but your starting paper will not be in the shape of a straingt strip. Consider this in the challenge at the end!

Tetraflexagons (type 4#n):

Here, four sub-squares make a larger square, and pinch flex just folds it down the middle to expose new sides.

See a handout with examples from another article by Martin Gardner
Look for the article of that title in his second book of diversions, available at:
https://bobson.ludost.net/copycrime/mgardner/gardner02.pdf

Conclusion: evaluating this as a subject for study

- Is this "interesting mathematics"?
- Is it productive?
- How can we generalize this?

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As a class, we sort of decided these were fun, but were not convinced to pursue them too much further.

Some generalizations include other shapes, and other folding patterns besides pinch flexes. For extensive examples, see:
http://loki3.com/flex/

One generalization I have not found papers about (but consider interesting) is generalizing these to higher dimensions. We discsussed this briefly in class.

Challenge for next time

(A prize will be awared for a correct answer):

Are there additional examples of 6#6 besides the one we built?

(Hint: consider the appending to Gardner's haxaflexagon chapter for recipes, and consider whether graphs (or 2-simplexes) are equivalent)