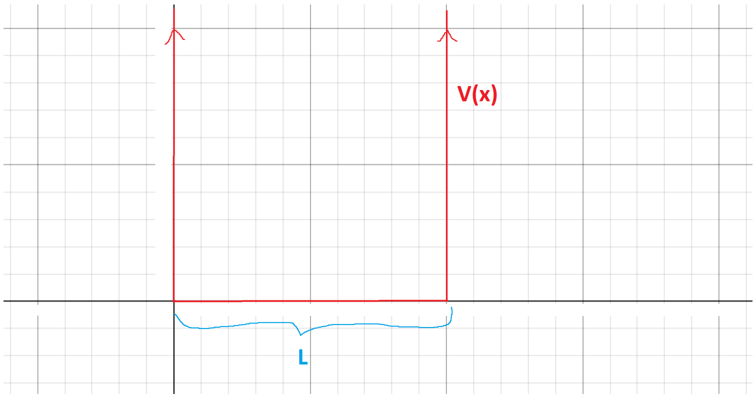
Worksheet solutions

1. [easier] The "particle in a well" system is defined to have the potential energy function

$$V(x) = \begin{cases} 0 & 0 \le x \le L \\ +\infty & \text{else} \end{cases}$$

where $L > 0$. Graph the function $V(x)$ and find what potential energies the particle would have at $x = \frac{L}{2}, L, L+1$.

Solution: The graph of $V(x)$ looks something like this...



Evaluating $V(x)$ at $x = \frac{L}{2}, L, L+1$, we find

$$V(L/2) = 0, \quad V(L) = 0, \quad V(L+1) = \infty$$

2. [easier] If a particle is in the "particle in a well" system (with $V_0 = 0$), and is in the first energy state, then the particle has the energy

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2}$$

Suppose out particle is an electron with mass $m_e = 9.1 \times 10^{-31}$ kg in a well with length $L = 10^{-9}$ m and $\hbar = 1.05 \times 10^{-34}\,\mathrm{m^2 kg/s}$. Using a calculator, find E_1 for our electron.

Solution: Plugging in all the values into the formula, we get

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.05 \times 10^{-34}\,\mathrm{m^2 kg/s})^2}{2(9.1 \times 10^{-31}\,\mathrm{kg})(10^{-9}\,\mathrm{m})^2} = 5.98 \times 10^{-20}\,\mathrm{kg\,m^2/s^2}$$

3. [easier – medium] Consider the "particle in a well" system at the three lowest energy levels: $n = 1, 2, 3$. Sketch the function $p(x)$ (defined in Eq. 1) for each of these values of n.

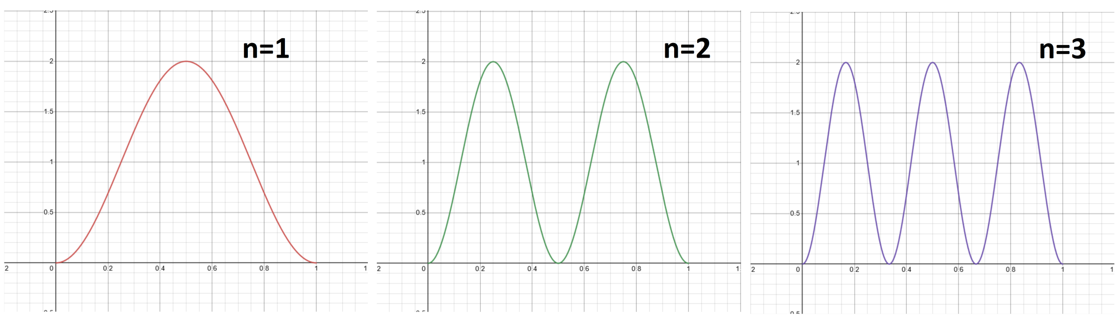
Solution: For the particle in the well, the wavefunction was found to be

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) & 0 \le x \le L \\ 0 & \text{else} \end{cases}$$

Therefore the probability density function for each of these wavefunctions, $p_n(x)$, is

$$p_n(x) = |\psi_n(x)|^2 = \begin{cases} \frac{2}{L} \sin^2\left(\frac{n\pi}{L}x\right) & 0 \le x \le L \\ 0 & \text{else} \end{cases}$$

Plugging in values for $n = 1, 2, 3$, we get the following plots (taking $L = 1$ as an example). Note that each graph has n nodes, all graphs have $p(x) = 0$ for $x \le 0$ and $x \ge L$.



4. Consider the "particle in a well" system.

   (a) [medium – has integration] Suppose our particle is measured to be in the first energy level, $n = 1$. What is the probability that our particle is in the "left" half of the well given by $[0, L/2]$. Explain why this probability makes sense.

Solution: If the particle is measured in the first energy level, $n = 1$, this corresponds to the wavefunction

$$\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right)$$

and the probability density function

$$p_1(x) = \frac{2}{L} \sin^2\left(\frac{\pi}{L}x\right)$$

The probability that the particle is in the left half of the is found by integrating this probability density function over the range $[0, L/2]$:

$$\begin{aligned}
P\left(0 \le x \le \frac{L}{2}\right) &= \int_0^{L/2} p_1(x)\, dx \\
&= \int_0^{L/2} \frac{2}{L} \sin^2\left(\frac{\pi}{L}x\right)\, dx \\
&= \int_0^{L/2} \frac{2}{L}\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi}{L}x\right)\right] dx \\
&= \left[\frac{x}{L} - \frac{1}{2\pi}\sin\left(\frac{2\pi}{L}x\right)\right]\Bigg|_0^{L/2} \\
&= \frac{1}{2}
\end{aligned}$$

So the probability that the particle is in the left half of the well is 50%.

(b) [medium] For the particle in part (b), what is the probability it is in the right half of the well $[L/2, L]$? Explain why the probabilities from part (a) and (b) make sense.

Solution: Since the probability the particle is outside the well is 0% ($p(x) = 0$ for $x \le 0$ and $x \ge 0$), the particle also has a 50% probability of being in the right half of the well. It's reasonable that the particle has an equal chance of being in the first and second halves of the well since there is nothing physically distinct about either of the sides - the probability of the particle existing at any point inside the well is symmetric about the middle of the well (see the $n = 1$ case in 1(a)).

(c) [tricky – integration] Suppose there are now two particles in the well incapable of interaction, one in the $n = 1$ energy level, and the other in the $n = 3$ energy level. (It's important they are incapable of interaction so they can't change energy levels after observation). What is the probability that the particles will both be in the region $[L/4, L/2]$ simultaneously?

Solution: Label the $n = 1$ particle as particle 1 and the $n = 3$ particle as particle 2. The wavefunctions, and their corresponding probability density functions, for the $n = 1$ and $n = 3$ case were found in the previous parts:

$$\psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right), \quad \psi_3(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi}{L}x\right)$$

$$p_1(x) = \frac{2}{L}\sin^2\left(\frac{\pi}{L}x\right), \quad p_3(x) = \frac{2}{L}\sin^2\left(\frac{3\pi}{L}x\right)$$

The probability that particle 1 and particle 2 will be in the region $[L/4, L/2]$ simultaneously is the product of the probabilities of the two being in the region individually, i.e.

$$P\left(\frac{L}{4} \le x_1, x_2 \le \frac{L}{2}\right) = P\left(\frac{L}{4} \le x_1 \le \frac{L}{2}\right) \cdot P\left(\frac{L}{4} \le x_2 \le \frac{L}{2}\right)$$

This is because we assume the particles are noninteracting, and therefore these two probabilities are independent. Now we just need to calculate these two probabilities by integrating the appropriate probability density functions:

$$
\begin{aligned}
P\left(\frac{L}{4} \leq x_1 \leq \frac{L}{2}\right) &= \int_{L/4}^{L/2} p_1(x)\, dx \\
&= \int_{L/4}^{L/2} \frac{2}{L} \sin^2\left(\frac{\pi}{L} x\right)\, dx \\
&= \left[\frac{x}{L} - \frac{1}{2\pi} \sin\left(\frac{2\pi}{L} x\right)\right]\Bigg|_{L/4}^{L/2} \\
&= \frac{1}{4} + \frac{1}{2\pi} = \frac{2+\pi}{4\pi} \\
P\left(\frac{L}{4} \leq x_2 \leq \frac{L}{2}\right) &= \int_{L/4}^{L/2} p_3(x)\, dx \\
&= \int_{L/4}^{L/2} \frac{2}{L} \sin^2\left(\frac{3\pi}{L} x\right)\, dx \\
&= \int_{0}^{L/2} \frac{2}{L} \left[\frac{1}{2} - \frac{1}{2} \cos\left(\frac{6\pi}{L} x\right)\right]\, dx \\
&= \left[\frac{x}{L} - \frac{1}{6\pi} \sin\left(\frac{2\pi}{L} x\right)\right]\Bigg|_{L/4}^{L/2} \\
&= \frac{1}{4} + \frac{1}{6\pi} = \frac{2+3\pi}{12\pi}
\end{aligned}
$$

And so the probability both particles will be in the window is

$$
P\left(\frac{L}{4} \leq x_1, x_2 \leq \frac{L}{2}\right) = \left(\frac{2+\pi}{4\pi}\right)\left(\frac{2+3\pi}{12\pi}\right) = \frac{(2+\pi)(2+3\pi)}{48\pi}
$$

5. [medium - can be done without knowing much about differential equations] A central idea of quantum mechanics stems from the mathematical fact that scalar multiples of a solution to the Schrodinger equation gives other solutions. If we have a wavefunctions $\psi_(x)$ which is a solution to the Schrodinger Equation - Eqn. 3, then the wavefunction

$$
\Psi(x) = \alpha \psi(x)
$$

is also a solution for all real constants α. Show, by plugging into Eqn. 3, that if $\psi(x)$ satisfies the Schrodinger equation then $\Psi(x)$ also satisfies the Schrodinger equation as claimed above.

Solution: Suppose $\psi(x)$ is a wavefunction that satisfies the Schrodinger equation and α is a real constant. Since $\psi(x)$ satisfies the Schrodinger equation, we have

$$
\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)
$$

Where E is some real constants. Plugging, Ψ into the RHS Schrodinger eqn. 3 and simplifying, we get

$$\begin{aligned}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\Psi(x) &= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\alpha\psi(x)\\ &= \alpha\underbrace{\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi(x)}_{E\psi(x)}\\ &= E\alpha\psi(x)\\ &= E\Psi(x)\end{aligned}$$

So since there exists an E such that

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\Psi(x) = E\Psi(x)$$

namely, the same E which satifies the $\psi(x)$ Schrodinger equation, the wavefunction $\Psi(x)$ is a solution to the Schrodinger equation also.