BMC Fall 2022 - Intermediate I

Tiling a $2 \times n$ with dominoes

By Harry Main-Luu

I. Overarching question

Given a $2 \times n$ bar of chocolate, how many ways can we break it into n pieces of 2×1 (domino style) to share to n friends?

II. Warm-Ups

We will write the Fibonacci sequence in a strange way. We will use powers of a dummy variable x to keep track of the order. That is, each number in the sequence will be attached to a power of x, where the power tells us which term it is, as follow:
Instead of writing $1, 1, 2, 3, 5, ...$, we write

$$F(x) = 0x^0 + 1x^1 + 1x^2 + 2x^3 + 3x^4 + ... = \sum_{n=0}^{\infty} a_n x^n$$

We call $F(x)$ a generating function (because it looks like a function, and it generates the sequence we want?!) Then, what is the coefficient of x^5 in $F(x)$, what does it mean if you see the term $89x^{11}$ in F?
Another example: Write the sequence $1, 2, 3, 4, ...$ as a generating function!

III. Technical Preparation

We want to derive a formula for the number in front of the general power x^n in the expanded form of F. That number, called the coefficient, will be the *nth* Fibonacci number. Hopefully, this number will be a function of n.

1. Let $\phi_1 = \frac{\sqrt{5}+1}{2}, \phi_2 = \frac{\sqrt{5}-1}{2}$. Do the following calculations:

   a) $\phi_1 - \phi_2$

b) $\phi_1 \times \phi_2$

2. Perform the following calculation

$$\Big(\frac{1}{8-x} - \frac{1}{13-x}\Big)\frac{1}{8-13}$$

You might notice that the end result completely depends on the numbers $8, 13$. That should suggest that the same calculation can be carried out for any two numbers. Can you guess which two numbers we are going to use?

3. Decompose the following into the sum of two fractions with linear denominator:

a) $\dfrac{1}{x^2-2x-3}$

b) $\dfrac{1}{2x^2-5x+3}$

c) $\dfrac{1}{x^2-x-1}$

d) (Generalized) $\dfrac{c}{(x-a)(x-b)}$

e) for funsies: Calculate: $\displaystyle\sum_{n=1}^{7} \frac{1}{(n+1)(n+2)}$

4. Prove (or believe)

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... = 2$$

5. Prove (or believe)
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$
Does it match with the number we found earlier?

IV. Putting facts together

Calculate:
$F - xF - x^2F =$

Then, we have $F = \frac{x}{1-x-x^2}$.
From here, we perform a lot of mathemagic to get to the final formula:
$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

You should check the first few numbers to see if it works!

V. Exercises and Expansions

What is a recursive relation?
Does it have anything to do with why we manipulate our generating functions?
Find the generating function for the following problems. If possible, try to find a closed form for the number of ways to count the problems.

1. $a_{n+2} = 2a_{n+1} + 3a_n; a_0 = 1, a_1 = 1$

2. How many ways can we break the $2 \times n$ block of chocolate into pieces of 2×1, 1×2 and/or 2×2 blocks. Think of it as a chocolate bar with white chocolate on the top row, dark chocolate on the bottom row.

3. For all $n \in \mathbb{N}$, let a_n be the number of ternary sequences of length n with the property that product any two consecutive digits is 0. Find a closed formula for a_n.

   (Hint: Use a similar strategy as in the exercise above.)