Hall's Matching Theorem

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1 Warmup Problems

Problem 1 A group of one hundred students, with no two exactly the same height, were arranged in a square formation. In each of the ten rows, the shortest student raised his or her hand – of these students, John was the tallest. Then, in each of the ten column, the tallest student raised his or her hand; of these, Mary was the shortest. Who is taller, John or Mary?

Problem 2 Warmup Activity I need 13 volunteers. We deal out a full deck of 52 playing cards to the volunteers, each gets 4 cards. I'd like to have them each give me a card with a different rank (Ace, 2, 3, 4,. . . , J, Q, K). Is this always possible? Or can we find a way to give them out 4 cards each so that we can't get a different card from each person? Of course, if we *can* do it, they are left with 3 cards each. If so, can I *again* manage to get a different card from each person? And if so, could we do it when they each have only two cards left? (of course, if we get them down to one card each, we can surely finish).

Could we do the same game with four volunteers, thirteen cards each, and I want to get a different *suit* from each of them? (Clubs, Diamonds, Hearts, Spades)

Problem 3 I want to pick a *unique* integer from each set. Can we do it? (They didn't need to all have 3 elements in them, it was just easier to generate that way). What can go wrong? If it can't be done in some cases, how can we be sure?

First example

Set 1:	{6, 8, 10}	Set 2:	{4, 7, 8}	Set 3:	{1, 5, 9}	Set 4:	{4, 6, 8}	Set 5:	{2, 7, 10}
Set 6:	{1, 8, 9}	Set 7:	{1, 2, 7}	Set 8:	{2, 7, 10}	Set 9:	{1, 4, 10}	Set 10:	{1, 5, 6}

Second example

Set 1:	{1, 3, 5}	Set 2:	{4, 5, 10}	Set 3:	{2, 4, 9}	Set 4:	{1, 7, 10}	Set 5:	{2, 8, 9}
Set 6:	{3, 5, 8}	Set 7:	{4, 9, 10}	Set 8:	{2, 7, 9}	Set 9:	{3, 7, 9}	Set 10:	{1, 6, 8}

Third example

Set 1:	{3, 5, 9}	Set 2:	{3, 5, 6}	Set 3:	{4, 8, 10}	Set 4:	{5, 6, 7}	Set 5:	{2, 7, 8}
Set 6:	{1, 3, 5}	Set 7:	{1, 7, 9}	Set 8:	{4, 6, 9}	Set 9:	{1, 6, 9}	Set 10	{5, 7, 9}

Fourth example

Set 1:	{2,6, 8}	Set 2:	{5,6, 8}	Set 3:	{3, 5, 9}	Set 4:	{4, 7, 8}	Set 5:	{1, 9, 10}
Set 6:	{4, 6, 8}	Set 7:	{2, 4, 9}	Set 8:	{4, 5, 8}	Set 9:	{5, 7, 8}	Set 10:	{1, 2, 9}

2 Contest-style Problems once we've learned the theorem

Who is "marrying" whom?

Problem 4 In a $2n \times 2n$ chessboard, there are n rooks in each row and each column of the board. Show that there exist $2n$ rooks no two of whom are in the same row and same column. (Hint: we are "marrying" rows to columns.)

Problem 5 (Putnam 2012) A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

Problem 6 (Kazakhstan 2003) We are given two square sheets of paper with area 2003. Suppose we divide each of these papers into 2003 polygons, each of area 1. (The divisions for the two piece of papers may be distinct.) Then we place the two sheets of paper directly on top of each other. Show that we can place 2003 pins on the pieces of paper so that all 4006 polygons have been pierced.

Problem 7 (Canada 2006) In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

Problem 8 (Baltic Way 2013) Santa Claus has at least n gifts for n children. For $i \in \{1, 2, \ldots, n\}$, the ith child considers $x_i > 0$ of these items to be desirable. Assume that

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \leq 1$$

Prove that Santa Claus can give each child a gift that this child likes

3 A few theorems that follow from Hall's Matching Theorem

We *might* mention one or two of these today, though they would really require a second math circle session to really cover them. (And there are several other theorems we could add to this list!)

Theorem 1 (Kőnig's Matching Theorem) *Prove that if all the vertices of a bipartite graph have the same degree, then it has a perfect matching. (compare to the warmup problems)*

Wait! What's a bipartite graph? What does it mean for vertices to "have the same degree"? What's a perfect matching? (We might actually answer these questions today)

Theorem 2 (Birkhoff-von Neumann) *Any doubly-stochastic matrix may be represented as a convex combination of permutation matrices*

Wait! What's a doubly-stochastic matrix? What's a permutation matrix? What's a convex combination? And what on earth does this have to do with Hall's marriage theorem? [Hint: each row of the matrix is going to "marry" each column and we're doing to use that to get one of our permutation matrixes.]

Theorem 3 (Dilworth) *If every antichain in a (finite) partially ordered set has at most m elements, then the set may be partitioned into m chains.*

Wait! What's a partially ordered set? What's a chain? What's an antichain?

4 Hall's marriage theorem

Here's how it's traditionally stated. I find it a little cringe-inducting, but it's really a theorem about bi-partite graphs.

Theorem 4 (Hall's marriage theorem) *Given a collection of people containing n men and at least n women, the following two properties are equivalent:*

- *For* any *subset of the men [including the set of all n men], if the number of men in the subset is k, then there are at least k women in the collection known to at least one man in the subset.*
- *There is a way to match each man with a distinct woman he knows in the collection.*

Yuck. But does the mathematical idea make sense? Here's way to say it more abstractly.

4.1 Hall's marriage theorem – abstractly

Theorem 5 (Hall's matching theorem) *Given a collection of n sets of positive integers: $A_1, A_2, \ldots A_n$ (the sets are not necessarily distinct), the following two properties are equivalent*

- *For* any *subcollection of the n sets [including the subcollection of all the sets], if the number of subsets in the subcollection is k, then the union of those subsets contains at least k distinct integers.*
- *There is a way to pick a distinct integer from each set (i.e. the same integer is not chosen for two or more sets).*

Can you see that this is equivalent to the previous description? The "men" in the previous description are the indices (i.e. the subscripts) of the sets, and the "women" known to the ith "man" are the integers belonging to the set A_i.

4.2 Another even mathier way to say it

Definition 6 *A* bipartite graph *has two disjoint sets of vertices A and B such that every edge in the graph connects a vertex in set A to a vertex in set B.*

Further, for every set of vertices $X \subseteq A$ we can define the set

$$Y_X = \{y \in B \mid \textit{there's an edge in the graph from some element of } X \textit{ to } y\}.$$

Theorem 7 (Hall's marriage theorem) *If we have a finite bipartite graph as above then the condition that, for every subset of vertices $X \subseteq A$, we have $|X| \leq |Y_X|$ is equivalent to the condition that there is a subset of edges of the graph (aka a* matching*) that takes every vertex in A to a distinct vertex in B.*

Can you see that this is equivalent to the previous two descriptions?

5 Proof of the theorem

(I'm going to use the second way of expressing the theorem, with the sets $A_1, A_2, \ldots, A_n$) If we have trouble visualizing it, we might want to go back to the card example.)

The theorem says that two things are equivalent, but maybe we can immediately see why IF there's a way to select a distinct integer from each set, the weird condition on unions of subcollections must also be true. (can someone explain it during the circle?) So let's focus on the other direction, showing that when the weird condition is met, there must be a way to select a distinct integer from each set.

We'll use induction on n. The **base case**, when $n = 1$, is pretty immediate: There's exactly one set A_1 and it has at least one integer in it. If we have time, we might explicitly work out the case when $n = 2$ in class, too, though we don't *need* to do that for our proof.

5.1 The Inductive Step

For the inductive step, suppose we've managed to verify the result really is true for all the positive integers up to n. We'd like to use that to prove the result for $n + 1$. So we begin with a collection of sets $A_1, A_2, \ldots A_{n+1}$ that obey the weird condition on unions of subcollections, and we want to find a way to pick a distinct integer from each set.

So let's just look at the "last" subset, A_{n+1}. It contains at least one integer (why?), so let's just take some integer from it, let's call it x_{n+1} and see if we it is possible to select distinct integers *other* than $x_{n+1}$from all the other sets $A_1, A + 2, \ldots A_n$. If we can, great, we're done! But what if it simply isn't possible to do and we've painted ourselves into a corner?

5.2 What might go wrong

By the inductive hypothesis, that means that, after we removed x_{n+1} from all the sets $A_1, A_2, \ldots A_n$, the weird condition on unions of subintervals wasn't met. In other words, there must be a subcollection of these n sets, whose union has *fewer* integers than the number of sets in the subcollection. But if you restore x_{n+1} to the sets in that subcollection, their union would have *at least* as many integers as the number of sets in that subcollection. (do you see why?) So that means the union of the sets in that subcollection must have *exactly* the same number of integers as there are sets in the subcollection (do you see why?).

Let's call the sets in that subcollection $B_1, B_2, \ldots B_k$, and the remaining sets $C_1, C_2, \ldots C_{n+1-k}$. Since k is less than or equal to n, we can apply the induction hypotheses to the sets $B_1, B_2, \ldots, B_k$, so we can pick distinct representatives for each of them.

5.3 Out of the corner

If we then remove any occurences of those representatives from all of the sets $C_1, C_2, \ldots C_{n+1-k}$, we'd like to verify that *these* sets obey the weird union of subcollections property. That is, there will be at least j distinct integers in the union of any subcollection of j of these sets (for any j between 1 and $n + 1 - k$. But if some such subcollection of j of these sets had a union that was smaller than j, then the union of these sets with the sets $B_1, B_2, \ldots, B_k$ would have size smaller than $j + k$, which contradicts the conditions of the problem.

6 applying that theorem to the card activity

In the card puzzle we played with, each pile is being "married" to a rank; or, to describe the problem another way: each pile of cards represents a vertex in set A, and each rank of card represents a vertex in set B, and there's an edge between a pile and a rank if there's at least one card of that rank in the pile. By having each pile give us one card with a different rank, we are creating a matching between the pile-vertices and the rank-vertices.

In either case, since each pile has the same number of cards as remain for each rank, you can see by the pigeonhole principle that the union of k distinct piles must contain at least one card for at least k distinct ranks.