MATHEMATICAL "EVOLUTION"

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BMC INTERMEDIATE II

$p_2 \cdot p_5 \cdot p_{860}$

Warm-ups

1 *Breaking the Bar*. Start with a rectangular chocolate bar which is 6×8 squares in size. A legal move is breaking a piece of chocolate along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. Two players alternate turns. The person who makes the last legal move gets to eat all the chocolate. Find a strategy to win this game! What about the general case (the starting bar is $m \times n$)?

2 *In-and-out*. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 2022 minutes, could the room contain 1001 people?

3 *Putting Away Pool Balls*. Suppose you have an infinite number of pool balls, which each have a positive number written on it. For each integer label, there is an infinite supply of balls with that label.

You have a box which contains finitely many such balls. (For example, it may have six #3 balls and twelve #673 balls and a million #2 balls.) Your goal is to empty the box. You may remove any *single* ball you want at each turn. However, whenever you remove a ball, your little brother then adds more balls to your box with smaller labels. Your brother can put *any finite number* of balls in, as long as they have a smaller label value. For example, if you remove one #3 ball, your brother can replace it with 50 #2 balls and 30 trillion #1 balls.

But If you remove a #1 ball, then your brother cannot do anything, since there are no balls with lower-valued labels. Is it possible to empty the box in a finite amount of time?

Medium to Hard

4 *A Darwinian Struggle*. At time $t = 0$ minutes, a virus is placed into a colony of 2,022 bacteria. Every minute, each virus kills one bacterium apiece, after which all the bacteria and viruses divide in two. For example, at $t = 1$, there will be $2021 \times 2 = 4042$ bacteria and 2 viruses. At $t = 2$, there will be 4040×2 bacteria and 4 viruses, etc. Will the bacteria be driven to extinction? If so, when will this happen?

5 *Frogs*. Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the "leapee" is at the midpoint of the line segment whose endpoints are the starting and ending position of the "leaper." Will a frog ever occupy the vertex of the square that was originally unoccupied?

6 *Make it monochrome*. Start with a row of 20 playing cards with alternating orientation: The first card is face up, the next one is face down, etc. Your goal is to turn all the cards face up, but only by making moves according to this rule: at each move, you may select one or more *consecutive* cards and flip each of these cards over (without changing their order). For example, you could pick the first 4 cards and flip them, and now the orientations would be down, up, down, up, up, ... You would like to do this in as few moves as possible.

(a) Explain how to do it in 10 moves.

(b) Can you improve (a)? Explain.

7 Initially, we are given the sequence $1, 2, \ldots, 100$. Every minute, we erase any two numbers u and v and replace them with the value $uv + u + v$. Clearly, we will be left with just one number after 99 minutes. What can you say about this number?

Hard and Harder

8 *Maxim Kontsevich's Puzzle*. A penny is placed at $(0,0), (0,1)$, and $(1,0)$. At each turn, you can remove a penny, but then you must replace it with a penny one unit above it and one unit to the right, provided that both of these spots are empty. Is it possible, in finitely many moves, to ensure that the three original locations are all free of pennies?

9 *How Rich Can You Get?* Six boxes are arranged in a row, each containing one penny.

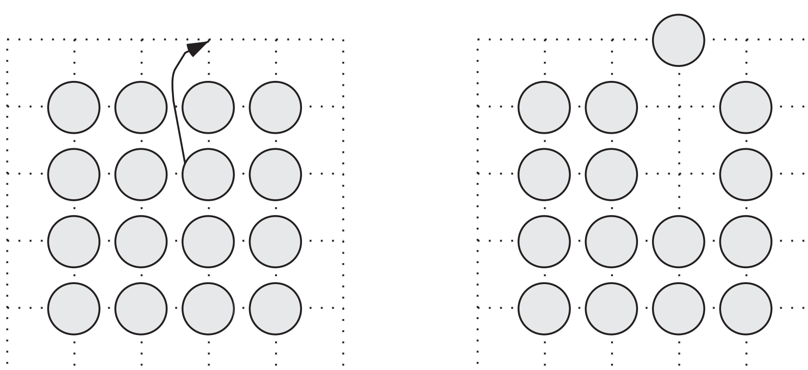
(a) Rule A says: if a box contains at least one penny, you can remove that penny, and move two pennies to the box immediately to the right. Notice that you cannot apply this rule to pennies in the rightmost box, since they have nowhere to go. By applying rule A, how many pennies can you accumulate in the rightmost box?

(b) Rule B says: if a box contains at least one penny, and there are two boxes to the right of it, you may remove a penny from this box, and then exchange the pennies in these two boxes. For example, if three boxes contains, in order from left to right, $6,0,3$, then after you apply rule B, the boxes now contain, in order from left to right: $5,3,0$.

If you start with the six boxes as above, each containing one penny, and you are allowed to apply rules A and B (whichever you want, whenever you can apply them), how much money can you accumulate in the rightmost box?

10 *Bulgarian Solitaire*. Start with n beans (or cards, or ducks, etc.), placed in piles. Each turn consists of removing one bean from each pile and making a new pile with the beans you took. For example, if you start with $n = 6$ beans, in piles of size $4, 1$, and 1, on the next move you will have two piles of size 3, and after that you will have three piles of size 2, etc. Investigate what happens with different starting values of n and different configurations of piles.

11 *Conway's Checker Problem*. Put a checker on every lattice point (point with integer coordinates) in the plane with y-coordinate less than or equal to zero. The only legal moves are horizontal or vertical "jumping." By this we mean that a checker can leap over a neighbor, ending up two units up, down, right, or left of its original position, provided that the destination point is unoccupied. After the jump is complete, the checker that was jumped over is removed from the board. Here is an example.



before jump *after jump*

Is it possible to make a finite number of legal moves and get a checker to reach the line $y = 5$?