

Polygon Decomposition: Part II - The Squaring Problem

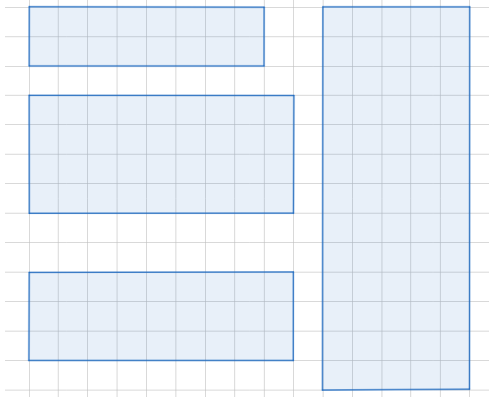
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(Continued from last time)

For the first two sessions, we will focus on one type of problem:

Can we cut a certain shape into finitely many pieces and then glue it back into a perfect, beautiful square?

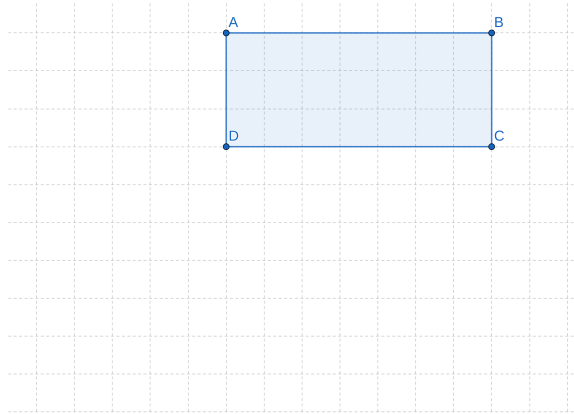
1. Let's start with the closest shape to a square, which, in this case, is a rectangle.



- a) You should notice by now that some rectangles are harder than others. Maybe we should ask which ones we can cut and glue back into a square and which ones we can not?

The most satisfying yet annoying answer is:

- b) Then how are we going to cut a random rectangle?



The key to this decomposition is the construction of the geometric mean of two numbers. In fact, this shows that we can construct the square root of a lot of numbers! Let's explore...

I. The Square Root and Its Constructibility

Definition I.1. Given a positive real number a , we define the square root of a , denoted by \sqrt{a} , to be the side length of the square with area a .

Examples and Exercises:

a) Find $\sqrt{1}$, $\sqrt{4}$, $\sqrt{13225}$.

b) True or False: Because $(-3) \times (-3) = 9$, we say that $\sqrt{9} = -3$.

c) True or False: For *any* x , $\sqrt{x^2} = x$

d) How should we write $\sqrt{x^2}$ as a function?

I.1. Practical Applications

Given a unit length below. Can you construct/draw the following lengths?

- $\sqrt{2}$, $\sqrt{3}$.
- \sqrt{n} , for any $n \in \mathbb{N}$.
- What about $\sqrt{\sqrt{2}}$?



I think we are ready to formulate the first big lemma! :)

Lemma I.1. If a positive length a is constructible, then \sqrt{a} is also constructible.

Proof. By construction, we construct the geometric mean of a and 1. □

Then, are there numbers we cannot construct? The answer is yes. When you do some more algebra, we can explain why such a number as $\sqrt[3]{2}$ is not constructible. Other famous inconstructible numbers include π, e . It took a long time before mathematicians figured out why this is the case! Constructions started in Ancient Greece, but only until a few thousand years later that we finally proved that we cannot construct those special numbers. In the interest of time, we will leave this discussion for the future (perhaps in only a few years instead of a few thousand!).

II. The Geometric Meaning of Averages

Before we explore the geometric meaning of the infamous averages, we need to discuss a new notion:

Definition II.1. A *locus* is a collection of points that share the same property, usually with reference to some fixed object(s).

Examples: circles := locus of points (on the same plane) equidistant to a fixed point (the center).

Exercise: Try to describe the following object using locus definition: perpendicular bisector of a segment, angle bisector of an angle, parabola, ...

- a) Those who are familiar with the Arithmetic Mean of two numbers (generally referred to as “the average”) might remember that it represents the midpoint of that segment. We can explore this concept in the sense of invariances (properties that don’t change).
 - i. On the real line, mark $A = (3)$ and $B = (7)$, where is their average (A.M.)?
 - ii. What is the locus of points with the property that the sum of its distance to A and B is 4?
 - iii. What is special about their arithmetic mean in this locus?

- b) Is there an analogue in Geometric Mean? We know that the Geometric Mean has to do with area and rectangles, so let’s start with a rectangle $ABCD$ of area 4 with a fixed vertex A at the origin ([click here](#) to see illustration).
 - i. If we want to fix the area at 4, and only allow B to move on the x -axis, D to move on the y -axis, what is the locus of C as the rectangle varies? This will require some advanced algebra! We can experiment!

ii. What is special about the G.M. of AB, CD ?

c) Expansion:

- i. Draw the diagonal BD of the rectangle. As C varies on our special curve, what do you think BD travel on?
- ii. Let's guess a few special "invariances" that seem reasonable and test them!

iii. What about the midpoint of the diagonal? Is it traveling on a special curve? Can we generalize this? Does this special curve depend on the area 4?

