Counting things up to symmetry
Berkeley Math Circle 2022 Feb 8

Examples of counting problems:
(1) In how many ways can one make a pizza with one of 2 (or 10) ingredients on each of the 6 slices?
(2) How many ways to make a necklace with 6 beads of 2 (or 10) colors?
(3) How many ways to place 8 rooks on a chessboard so no 2 attack each other?
(4) How many different graphs on 4 vertices are there?
(5) In how many ways can one color the faces of a cube with 2 colors? What about 10 colors?

In each case the main problem is we have to take symmetry into account: two apparently different ways may be the same under a symmetry. For small cases we can do this by hand (try the cube problem with 2 colors) but with more colors we need a systematic method.

The solution is given by the following formula (credited to Cauchy, or Frobenius, or Burnside, or Polya):

$(n_1+n_2+...n_g)/g$

where g is the number of symmetries and n_k is the number of solutions invariant under symmetry number k.

Pizza problem: There are 6 symmetries, as one can rotate the pizza.
We have n1=10^1, n2=10^2, n3=10^3, n4=10^2, n5=10^1 n6=10^6. So the total number of pizzas is 1001220/6=166870

Necklace problem. This differs from the pizza problem as we can also flip the necklace upside down in 6 ways. There are now 12 symmetries. In addition to the ones above there are 3 flips that fix 2 opposite beads, and 3 more that do not. We get n7=n8=n9=10^4 and n10=n11=n12=$10^{3.}$.
So the number of necklaces is 1034220/12=86185

Exercise: If p is an odd prime, find a formula for the number of necklaces with p beads and n colors. Why is this easier when p is a prime? Why does this formula not work when p=2?

Rook problem: There are 8 symmetries: 4 rotations and 4 reflections.
Count the number of fixed points for each of the 8 symmetries acting on the 8! arrangements of non-attacking rooks:

Identity: 8.7.6.5.4.3.2.1 =40320

2 reflections in a vertical or horizontal line: 0

2 reflections in a diagonal line: 1+8.7/2 + 8.7.6.5/2^2.2+8.7.6.5.4.3/2^3.3! + 8.7.6.5.4.3.2.1

$/2^4.4!=764$

2 quarter rotations: 6.2=12

1 half rotation: 8.6.4.2=384

Total (40320+2.0+2.764+2.12+1.384)/8 = 5282

Cube problem: $(1*10^6 + 8*10^2 + 6 * 10^3 + 6*10^3 + 3*10^4)/24$
as the cube has 24 symmetries:
1 identity with 6 orbits on faces
8 rotations by 1/3 with 2 orbits
6 rotations by 1/2 revolutions with 3 orbits
6 rotations by 1/4 revolution with 3 orbits

3 rotations by 1/2 revolution with 4 orbits.

Exercise: Do the same calculation, replacing a cube with a dodecahedron (there are now 60 possible rotations).

Graph problem for 4 points:

1 identity with 64 graphs

6 2-cycles with 16 graphs

8 3-cycles with 4 graphs

3 2.2 cycles with 16 graphs

6 4 cycles with 4 graphs.

Total (64+6*16+8*4+3*16+6*4)/24=11

Find them all and do the case of 5 vertices.

Exercise: Use the counting formula to show there are 2 alkanes with 4 carbon atoms (of course this is easier to check directly!) Here an alkane on n carbon atoms is a connected graph on n points with n-1 edges.

Why does the counting formula work? Look at the number of pairs (g,x) where g is a symmetry and x is something fixed by the symmetry. One the one hand if we count over g we find this number is equal to (n1+...ng). On the other hand, if we count over orbits of x, we get h elements of g for each element of an orbit of size g/h. So each orbit contributes g to the sum, and the sum is g times the number of orbits.