Berkeley Math Circle

February 2022

BEING AVERAGE

What does it mean to be an average American?

In fact, America is having all kind of problem in defining "average".

Before, being "average" means being "middle class".

A = $1000

B = $2000

C = $2000

D = $2000

E = $2000

F = $3500

G = $5000

H = $6000

I = $8500

J = $10000

A = \$1000

B = \$2000

C = \$2000

D = \$2000

E = \$2000

F = \$3500

G = \$5000

H = \$6000

I = \$8500

J = \$10000

Using our common sense of average, the average income of these 10 persons is:

(1000+2000+2000+2000+2000+3500+5000+ 6000+8500+10000) ÷ 10 = \$4200

A = \$1000	F = \$3000
B = \$2000	G = \$3500
C = \$2000	H = \$9000
D = \$2000	I = \$9500
E = \$2000	J = \$10000

Our common way of finding average gives us an average income of:

$$(1000+2000+2000+2000+2000+3000+3500+9000+9500+10000) \div 10 = \$4300$$

which totally distorted real life situation since 70% of the people earned less than that average.

The technical term for that kind of average is “Mean”. In this situation, two other kinds of “average” represent the real situation better. Median and Mode.

A = \$1000	F = \$3000
B = \$2000	G = \$3500
C = \$2000	H = \$9000
D = \$2000	I = \$9500
E = \$2000	J = \$10000

Median: the number located at exactly the middle. In this case, Median = (2000+3000)÷2 = \$2500. Half of the population earned less and the other half earned more. This looks better than using “Mean”. However, that number still does not look right.

The technical term for that kind of average is "Mean". In this situation, two other kinds of "average" represent the real situation better. Median and Mode.

A = $1000	F = $3000
B = $2000	G = $3500
C = $2000	H = $9000
D = $2000	I = $9500
E = $2000	J = $10000

Mode: the number that appeared the most number of times. In this case, Mode = $2000.

A = \$1000	F = \$3000
B = \$2000	G = \$3500
C = \$2000	H = \$9000
D = \$2000	I = \$9500
E = \$2000	J = \$10000

Mean: \$4300

Median: \$2500

Mode: \$2000

A = $1000

B = $1000

C = $2000

D = $2000

E = $2000

F = $2500

G = $2500

H = $3000

I = $200000

J = $300000

Mean: $5160

Median: $2250

Mode: $2000

For positive real numbers a, b, and c, prove:

$$(a+b)(b+c)(c+a) \geq 8abc$$

For $n \geq 1$, prove:

$$n! \leq \left(\frac{n+1}{2}\right)^n$$

When $x > 0$, find the minimum for

$$y = x + \frac{1}{x}.$$

Minimize

$$f(x) = \frac{9x^2(sinx)^2+4}{xsinx}$$

Arithmetic Average (Mean)

AM

Examples:

(1) 5 students in a class with test scores of 85, 70, 95, 82, and 68. What is their average (mean) score?

(2) If another class of 5 students has test scores of 100, 100, 100, 100 , and 0, their average test score is:

Examples:

(1) 5 students in a class with test scores of 85, 70, 95, 82, and 68. What is their average (mean) score?

$$\frac{85+70+95+82+68}{5} = 80$$

(2) If another class of 5 students has test scores of 100, 100, 100, 100 , and 0, their average test score is:

$$\frac{100+100+100+100+0}{5} = 80$$

We call this type of averages or means "Arithmetic Mean".

$$\text{AM:} \quad \frac{a+b}{2} \quad \text{or} \quad \frac{a_1+a_2+a_3+\ldots+an}{n} = 80$$

Geometric Average (Mean)

GM

(3) A product priced at $100 has two successive price markups with one 8% and then 18%. The arithmetic mean of these two markups is

$$\frac{8\% + 18\%}{2} = 13\%$$

Now, 8% markup on a $100 product means its price goes from $100 to ($100)×(1.08) = $108.

And an 18% markup after that means its price goes from $108 to ($108)×(1.18) = **$127.44**.

However, $127.44 is not the same as having an average markup of 13% twice.

($100) ×(1.13) ×(1.13) = $127.69 or one can say that

$$\mathbf{1.2744 = (1.08)\times(1.18) \neq (1.13)\times(1.13) = 1.2769}$$

What would be a good "average" or "mean" to better represent the average markup?

($100) × (1.08) × (1.18) = **$127.44**.

Consider $[(\$1.08) \times (1.18)]^{1/2} = [1.2744\ldots]^{1/2} = 1.12889\ldots$

($100) × (1.12889…) × (1.12889…) = **$127.44** and

So, one can say that 12.889…% is the average of 8% and 18%.

In fact, this type of average is called "Geometric Average" or "Geometric Mean" – GM.

AM of 8% and 18% = 13%

GM of 8% and 18% = 12.889…%

(4) Suppose you invest your money in a product that has the following returns: First year +20%, Second year –20% (a loss), Third year +40%, Fourth year –40% (a loss), Fifth year +50%. What would be this product's annual average return during these 5 years?

AM $= \frac{20\% - 20\% + 40\% - 40\% + 50\%}{5} = 10\%$

GM $= [(1.20) \times (0.80) \times (1.40) \times (0.60) \times (1.50)]^{1/5} = 1.03879\ldots$

or increased by 3.879…% per year.

CHECK:

Actual: $(\$100) \times (1.20) \times (0.80) \times (1.40) \times (0.60) \times (1.50)] =$ **\$120.96**

Use AM: $(\$100) \times (1.10) \times (1.10) \times (1.10) \times (1.10) \times (1.10)] =$ **\$161.05**

Use GM: $(\$100) \times (1.03879\ldots)^5 =$ **\$120.96**

Will show you later that **AM** always yields larger result than **GM**.

GM: $(a \times b)^{1/2}$ or $(a_1 \times a_2 \times a_3 \times \ldots \times a_n)^{1/n}$

Harmonic Average (Mean)

HM

(5) If you are driving 2 sections of a road in the same direction (100 miles each section) with a constant speed of 50 miles per hour in the first section and 25 miles per hour in the second section, what is your "average" speed for the entire road so that you will arrive in the same amount of time?

50 mph | 25 mph
100 miles | 100 miles

First section will take 100 miles/50 mph = 2 hours

Second section will take 100 miles/25 mph = 4 hours

Together, this trip took 2 hours + 4 hours = 6 hours in total.

Therefore, the average speed should be:

(200 miles) ÷ (6 hours) = **33.33… mph**.

AM: (50 mph + 25 mph)/2 = 37.5 mph

GM: $(50 \text{ mph} \times 25 \text{ mph})^{1/2} = 35.355\ldots$ mph

Neither one yields the correct answer.

Harmonic Mean (HM): 2 ÷ (1/50 + 1/25) = 2 ÷ (3/50) = **33.33… mph**.

HM: $2 \div (1/a + 1/b)$ or $n \div (1/a_1 + 1/a_2 + 1/a_3 + \ldots + 1/a_n)$

AM: $\frac{a+b}{2}$, **GM:** $(a \times b)^{1/2}$, **HM:** $2 \div (1/a + 1/b)$

Relationships Between AM, GM, and HM

$$(a - b)^2 \geq 0$$

$$(a^2 - 2ab + b^2) \geq 0$$

$$(a^2 + b^2) \geq 2ab$$

$$a^2 + 2ab + b^2 \geq 4ab$$

$$(a + b)^2 \geq 4ab$$

$$(a + b) \geq 2 \times (a \times b)^{1/2}$$

$$\frac{a+b}{2} \geq (a \times b)^{1/2}$$

So, **AM** $\geq$ **GM** Equality holds when $a = b$.

AM: $\frac{a+b}{2}$, **GM:** $(a \times b)^{1/2}$, **HM:** $\frac{2}{\frac{1}{a}+\frac{1}{b}}$

Relationships Between **AM**, **GM**, and **HM**

$$(a - b)^2 \geq 0$$
$$(a^2 - 2ab + b^2) \geq 0$$
$$(a^2 + b^2) \geq 2ab$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a + b)^2 \geq 4ab$$
$$1 \geq 4ab/(a + b)^2$$
$$a \times b \geq 4a^2b^2/(a + b)^2$$
$$(a \times b)^{1/2} \geq 2ab/(a + b)$$
$$(a \times b)^{1/2} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

So, **GM** $\geq$ **HM** equality holds when $a = b$.

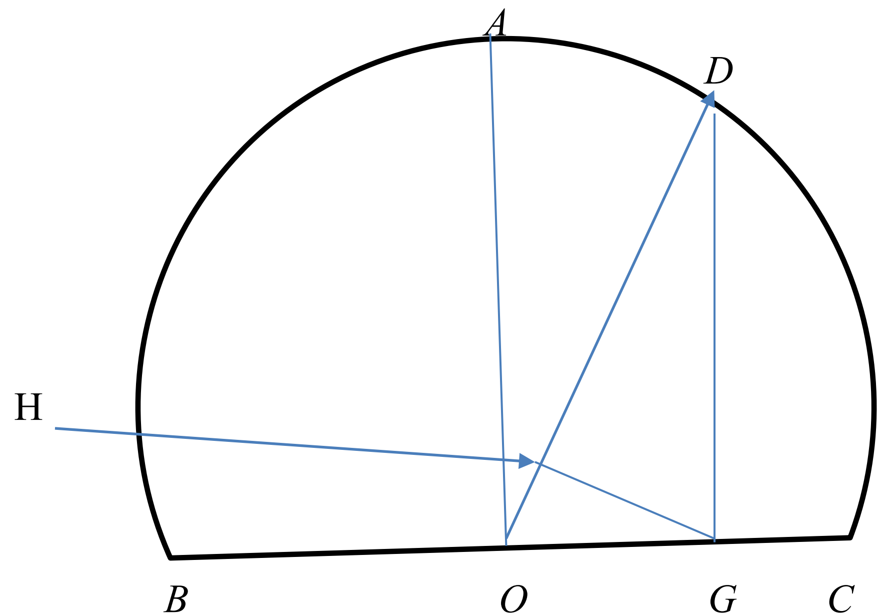
Therefore, **AM** $\geq$ **GM** $\geq$ **HM**

Equalities hold when $a = b$.

AM: $\frac{a+b}{2}$, GM: $(a \times b)^{1/2}$, HM: $2 \div (1/a + 1/b)$

Geometric Relationships Between AM, GM, and HM

Semi-circle of diameter $(a+b)$ or radius $(a+b)/2$.



Let $BG = a$ and $GC = b$. $GD = (OD^2 - OG^2)^{1/2}$. $\Delta OGD \sim \Delta GHD$.

Then **AM** $= AO = BO$, **GM** $= GD$, **HM** $= HD$

Therefore, **AM** $\geq$ **GM** $\geq$ **HM**

Equalities hold when $a = b$.

AM: $\frac{a+b}{2}$, **GM:** $(a \times b)^{1/2}$, **HM:** $2 \div (1/a + 1/b)$

Here is an interesting way to prove **GM ≤ AM**

We have already proved the case of $n = 2$.

$n = 4$. $\sqrt[4]{a_1 a_2 a_3 a_4} = \sqrt{\sqrt{a_1 a_2}\sqrt{a_3 a_4}}$

$\leq \frac{\sqrt{a_1 a_2}+\sqrt{a_3 a_4}}{2}$ n=2 case for **GM ≤ AM**

$\leq \frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2}$ n=2 case for **GM ≤ AM**

$\leq \frac{a_1+a_2+a_3+a_4}{4}$

This method can also be used to prove the cases where $n = 2^m$ by using mathematical induction.

$n = 2^m$. $\sqrt[n]{a_1 a_2 a_3 \ldots a_n} \leq \frac{a_1+a_2+a_3+\cdots+an}{n}$.

Now for general n. Let $n+r = 2^m$ and let

$A = \frac{a_1+a_2+a_3+\cdots+an}{n}$. Then, for r number of A's,

$$\sqrt[n+r]{a_1 a_2 a_3 \ldots a_n AA \ldots A} \leq \frac{a_1+a_2+a_3+\cdots+an+A+A+\cdots A}{n+r}$$

$$(a_1 a_2 a_3 \ldots a_n A^r)^{\frac{1}{n+r}} \leq \frac{nA+rA}{n+r} = A$$

$$(a_1 a_2 a_3 \ldots a_n A^r) \leq A^{n+r}$$

$$(a_1 a_2 a_3 \ldots a_n) \leq A^n$$

$$\sqrt[n]{a_1 a_2 a_3 \ldots a_n} \leq \frac{a_1+a_2+a_3+\cdots+an}{n}$$

So, **GM** $\leq$ **AM** or $\sqrt[n]{b_1 b_2 b_3 \ldots b_n} \leq \frac{b_1+b_2+b_3+\cdots+b_n}{n}$.

Let $b_k = 1/a_k$ for any $0 \leq k \leq n$. Then

$$\sqrt[n]{1/(a_1 a_2 a_3 \ldots a_n)} \leq \frac{1/a_1+1/a_2+1/a_3+\cdots+1/an}{n}$$

$$\frac{n}{1/a_1+1/a_2+1/a_3+\cdots+1/an} \leq \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

HM $\leq$ **GM**

Therefore, **AM** $\geq$ **GM** $\geq$ **HM**

Equality holds when $a_1=a_2=a_3= \ldots =a_n$

Consequence:

AM × **HM** = **GM**^{2}

$$(a + b)/2 \times 2/(1/a + 1/b) = (a + b)/2 \times 2ab/(a + b)$$
$$= ab$$

This means that **GM** is the Geometric Mean of **AM** and **HM**.

Arithmetic Mean is best used in situations where:

(1) the data are not skewed (no extreme outliers)

(2) the individual data points are not dependent on each other.

Geometric Mean is best used whenever the data are inter–related as in examples when discussing returns on investment or interest rates.

Harmonic Mean is best used when there is a large population where the majority of the values are distributed uniformly but where there are a few outliers with significantly large values.

Take an example of 14 numbers:

35 35 35 35 35 35

40 40 40

45 45

48

50

150

The arithmetic mean of these 14 numbers is 47.7 while the harmonic mean is approximately 41.

In this case, the harmonic mean provides a better picture of the true average of these numbers. Discounting the outlier 150, the arithmetic mean is only 39.8 and the harmonic mean is about 39.2.

Examples:

(1) A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per foot, while the fence for the other three sides costs \$1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is that minimum expense?

(1) A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per foot, while the fence for the other three sides costs \$1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

x = length along highway.

Cost = (\$2)*x + (\$1)[x+2(60000/x)] = $3x$ + 120000/x.

AM–GM Inequality implies:

$$\text{Cost} \geq 2\sqrt{(3x)\left(\frac{120000}{x}\right)} = 2\sqrt{360000} = 2(600) = \$1200.$$

Since the cost is \$1200 when x = 200 feet, cost is minimum at \$1200 which yields a 200 ft. × 300 ft. plot.

No Calculus is needed.

(2) Minimize $f(x) = \frac{9x^2(sinx)^2+4}{xsinx}$.

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$f(x) = 9x\text{sin}x + \frac{4}{xsinx}$. Apply AM–GM Inequality,

$$f(x) \geq 2\sqrt{9xsinx(\frac{4}{xsinx})} = 2\sqrt{9(4)} = 12.$$

Minimum of 12 can be obtained when

$$9x\text{sin}x = \frac{4}{xsinx} \quad \text{or} \quad x\text{sin}x = 2/3.$$

(3) For positive real numbers a, b, and c, prove:

$$(a+b)(b+c)(c+a) \geq 8abc$$

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By AM–GM,
$$a+b \geq 2\sqrt{ab}$$
$$b+c \geq 2\sqrt{bc}$$
$$c+a \geq 2\sqrt{ca}.$$
$$(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})$$
$$= 8abc$$

(4) When $x > 0$, find the minimum for

$$y = x + \frac{1}{x}.$$

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$$y = x + \frac{1}{x}.$$

$$x + \frac{1}{x} \geq 2\sqrt{(x)\left(\frac{1}{x}\right)}$$

$$= 2$$

Equality holds when $x = 1$.

(5) When $x > 0$, find the minimum for

$$y = \frac{x}{1+x^2}.$$

(5) When $x > 0$, find the maximum for

$$y = \frac{x}{1+x^2}.$$

$$\frac{x}{1+x^2} = \frac{1}{\frac{1}{x}+x} \leq ½[\sqrt{(x)\left(\frac{1}{x}\right)}\,] = ½.$$

Equality holds when $x = 1$.

(6) When $x \geq 0$, find the maximum for

$$y = (1-2x)(1+x)^2.$$

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$$y = (1-2x)(1+x)^2.$$

$$(1-2x)(1+x)^2 \leq \left[\frac{(1-2x)+(1+x)+(1+x)}{3}\right]^3$$

$$= 1$$

Equality holds when $x = 0$.

(7) When $x \geq 0$, find the maximum for

$$y = (1-x)(1+x)^2.$$

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$$y = (1-x)(1+x)^2.$$

$$(1-x)(1+x)^2 \leq \left[\frac{(1-x)+(1+x)+(1+x)}{3}\right]^3$$

$$= \left[\frac{3+x}{3}\right]^3$$

The technique from (8) does not work. We need a constant.

(7) When $x \geq 0$, find the maximum for

$$y = (1-x)(1+x)^2.$$

$$
\begin{aligned}
(1-x)(1+x)^2 &= (1/2)2(1-x)(1+x)^2 \\
&= (1/2)(2-2x)(1+x)(1+x) \\
&\leq (1/2)\left[\frac{(2-2x)+(1+x)+(1+x)}{3}\right]^3 \\
&= (1/2)\left[\frac{4}{3}\right]^3 = 32/27
\end{aligned}
$$

Equality holds when $x = 1/3$.

(8) In packaging a product in a can the shape of right circular cylinder, various factors such as tradition and supposed customer preferences may enter into decisions about what shape (e.g. short and fat vs. tall and skinny) can might be used for a fixed volume. Note, for example, all 12 oz. soda have the same shape – a height of about 5 inches and a radius of about 1.25 inches. Why?

(8) For any cylindrical can of radius r and height h, what is the relationship between r and h to minimize the surface area (including the top and bottom) of the can with a fixed volume V.

(8) For any cylindrical can of radius r and height h, what is the relationship between r and h to minimize the surface area (including the top and bottom) of the can with a fixed volume V.

$V = \pi r^2 h$ Surface Area $S = 2\pi r^2 + 2\pi r h$

$$h = \frac{V}{\pi r^2} \qquad S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = 2\pi r^2 + 2\left(\frac{V}{r}\right)$$

$$= 2\pi r^2 + \frac{V}{r} + \frac{V}{r} \geq 3\sqrt[3]{2\pi r^2\left(\frac{V}{r}\right)\left(\frac{V}{r}\right)}$$

$$= 3\sqrt[3]{2\pi V^2}$$

Equality holds when $2\pi r^2 = \frac{V}{r}$ or $V = 2\pi r^3$

or $h = 2r$.

(9) Find the maximum area of a sector of a circle with fixed perimeter.

$P = 2r + r\theta = r(2+\theta)$ or $\theta = (P-2r)/r$.

$$A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}r^2(P-2r)/r = \tfrac{1}{2}r(P-2r)$$

$$= \tfrac{1}{4}(2r)(P-2r) = \tfrac{1}{4}\left[\sqrt{(2r)(P-2r)}\right]^2$$

$$\leq \tfrac{1}{4}\left[\frac{(2r)+(P-2r)}{2}\right]^2$$

$$= (P/4)^2$$

Equality holds when $2r = P - 2r$

$$2r = (2r+r\theta) - 2r = r\theta$$

So, the maximum is at $\theta = 2$ which is about $120°$ ($114.59°$)

Exercises:

(1) Find all three means for $a = 36$ and $b = 64$.

(2) A bus travels between cities A and B. From A to B, the bus has an average speed of 40 mph. On its way back, the average speed is 60 mph. What is the average speed of the bus for the entire round trip?

(3) For $x \geq 0$, prove: $1 + x + x^2 + x^3 + \ldots + x^{2n} \geq (2n+1)x^n$.

(4) For $n > 1$, prove: $n! < (\frac{n+1}{2})^n$.

(5) For positive numbers a , b, and c, prove:

$$(a+b+c)(a^2+b^2+c^2) \geq 9abc.$$

(6) For any positive numbers $a_1, a_2, a_3, \ldots, a_n$, prove:

$$a_1/a_2+a_2/a_3+a_3/a_4+ \ldots +a_{n-1}/a_n + a_n/a_1 \geq n$$

Exercises:

(7) Prove $\frac{x^2+5}{\sqrt{x^2+4}} > 2$ for any number x.

(8) Prove that if the product of n positive numbers is 1, then their sum must be at least n.

(9) For any positive numbers $a_1, a_2, a_3, \ldots, a_n$, prove:

$$(a_1+a_2+a_3+ \ldots +a_{n-1}+a_n)(1/a_1+1/a_2+1/a_3+ \ldots +1/a_n) \geq n^2$$

(10) For any positive numbers $a_1, a_2, a_3, \ldots, a_n$, prove:

$$(a_1{}^n+a_2{}^n+a_3{}^n+ \ldots +a_{n-1}{}^n+a_n{}^n) \geq n(a_1a_2a_3 \ldots a_{n-1}a_n)$$

(11) For positive numbers a, b, c, and d, prove:

$$\frac{1}{a+b+c}+\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b} \geq \frac{16}{3}\cdot\frac{1}{a+b+c+d}.$$

Exercises:

(12) For any two unequal positive numbers a and b, prove: $ab^n < (\frac{a+nb}{n+1})^{n+1}$.

(13) For any positive numbers x, y, and z, prove:

$x^4+y^4+z^4 \geq x^2y^2+y^2z^2+z^2x^2 \geq xyz(x+y+z)$.

(14) For any positive numbers a, b, and c where $a+b+c=1$, prove:

$(1/a - 1)(1/b - 1)(1/c - 1) \geq 8$.

(15) For any positive numbers a_1, a_2, a_3, b_1, b_2, and b_3 prove:

$(a_1/b_1 + a_2/b_2 + a_3/b_3)(b_1/a_1 + b_2/a_2 + b_3/a_3) \geq 9$.

(16) For positive numbers a, b, and c, prove:

$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

Exercises:

(17) For any positive numbers a_1, a_2, a_3, …, and a_n, let $S = a_1+a_2+a_3+ \dots +a_{n-1}+a_n$. Prove:

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n} \geq \frac{n^2}{n-1}.$$

(18) For any positive numbers a_1, a_2, a_3, …, and a_n, let $S = a_1+a_2+a_3+ \dots +a_{n-1}+a_n$. Prove:

$$(1+a_1)(1+a_2)\dots(1+a_n) \leq 1+S+S^2/2!+S^3/3!+\dots+S^n/n!.$$

(19) Show that the maximum area of a rectangular region with a given perimeter is a square.

(20) Prove the maximum area of a triangle with fixed perimeter is equilateral.

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WMTC (November of each year)

World Mathematics Team Championship

Started in 2010 at Beijing

2015 – Beijing, China

2016 – South Korea

2017 – Thailand

2018 – Bulgaria

2019 – Hong Kong (South Korea)

2020 – Online (2300 students)

2021 – Online (2700 students from about 30 countries/regions)

2022 – Either Online or in Hanoi, Vietnam

2023 – Doha, Qatar

FORMAT (5-Days Event)

Three Levels (Junior, Intermediate, Advanced)

Five Parts:

Individual Round 1 – Multiple Choice Problems. Focus on speed and precision (30 Points)

Individual Round 2 – Single Answers. Focus on logical reasoning and analytical skills, (40 Points)

Team Round – Six students work as a team to solve 14 problems. Some problems are related to answers to others. (70 Points)

Relay Rounds – Three rounds of relays. Two students form a subgroup. (60 Points)

Tie-Breaker Round – In case of ties, this round determines the top four

Many regions requested to have more practice and training. Starting 2022, WMTC will be offering two more events.

2022 will be the first year we offer PWMTC (Practice WMTC).

PWMTC has exactly the same format as WMTC. Each region runs its own event at their own site. Depends on the pandemic situation, regions can conduct their PWMTC event online or in-person. The date to do PWMTC will be from April 8 to April 19.

Dr. Brown of San Diego Math Circle is in charge of recruiting students and coordinating the PWMTC and WMTC events in USA (except for the group from New Jersey). If any of you are interested, you contact contact me at Quan.Lam@Berkeley.Edu and I will refer you to Dr. Brown.

Each summer, we are planning to have a WMTC “training” camp. One was planned in China for August, 2020. We have put off this event for 2020, 2021, and very likely 2022 because of the pandemic. Participating students for this training camp will attend 5 days of cultural/sightseeing tours and training sessions at no cost. We have high hope to do our first event in 2023.