BMC Advanced
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Comparing Infinities I

Here is a summary of what was discussed at the first meeting and additional exercises.

1. Some notation

   - $\mathbb{N} := \{1, 2, 3, 4 \dots\}$ the set of natural (counting numbers)
   - $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$ the set of integers
   - $\mathbb{Q} := \{m/n \,|\, m \in \mathbb{Z}, n \in \mathbb{N}\}$ the set of rational numbers (fractions)
   - $\mathbb{R}$ the set of real numbers
   - For a *finite* set X, we denote by $\#X$ the number of elements in X.

2. Let $f : X \to Y$ be a function from a set X to a set Y. We say that

   - f is *injective* (or an *injection*, or *one-to-one*) if it sends distinct elements of X to distinct elements of Y, i.e. $f(a) = f(b)$ implies $a = b$.
   - f is *surjective* (or a *surjection*, or *onto*) if the range of f is all of Y, i.e. for every $y \in Y$ there exists $x \in X$ such that $y = f(x)$.
   - f is *bijective* (or a *bijection*, or *one-to-one correspondence*) if it is both injective and surjective. [Such functions are also called *invertible*. Can you guess why?]

3. Let X and Y be two sets. We say that

   - X and Y have the *same cardinality* if there exists a bijection $f : X \to Y$;
     notation: $|X| = |Y|$.
   - the *cardinality of X is less than or equal to the cardinality of Y* if there is an injection $f : X \to Y$; notation $|X| \leq |Y|$.
   - the *cardinality of X is greater than or equal to the cardinality of Y* if there is a surjection $f : X \to Y$; notation $|X| \geq |Y|$.
   - the *cardinality of X is less than the cardinality of Y* if $|X| \leq |Y|$ and $|X| \neq |Y|$; notation $|X| < |Y|$.

4. Some simple (but important) facts

   - If X and Y are finite sets then
     - $|X| = |Y|$ if and only if $\#X = \#Y$.
     - $|X| \leq |Y|$ if and only if $\#X \leq \#Y$.
     - $|X| \geq |Y|$ if and only if $\#X \geq \#Y$.
   - If X is a finite set and Y is infinite, then $|X| < |Y|$.
   - $|X| \geq |Y|$ is equivalent to $|Y| \leq |X|$. [This is not a tautological statement! Prove it.]
   - If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$. [Prove this!]
   - If $|X| = |Y|$ and $|Y| = |Z|$, then $|X| = |Z|$. [Prove this!]

   [It is also true that $|X| \leq |Y|$ and $|X| \geq |Y|$ imply $|X| = |Y|$, but it is harder to prove.]

5. A set X is called *countably infinite* if $|X| = |\mathbb{N}|$, i.e. elements of X can be labeled by natural numbers. We say that X is *countable* if it is finite or countably infinite.

   We saw that countably infinite sets are the *smallest* among infinite sets. More precisely,

   - X is finite if and only if $|X| < |\mathbb{N}|$;
   - X is infinite if and only if $|X| \geq |\mathbb{N}|$;
   - If X is countable and $Y \subset X$, then Y is also countable.

6. Using our experience with the *Infinity Hotel* we established the following facts:

   - The union $X \cup Z$ of a countably infinite set X and a finite set Z is countably infinite.
   - The union $X \cup Y$ of two countably infinite sets is countably infinite.
   - If $X_1, X_2, X_3, \ldots,$ is a sequence of sets in which each X_k is countably infinite, then the union $X_1 \cup X_2 \cup X_3 \cup \ldots$ is also countably infinite.

     In particular, this shows that the sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

7. Exercises

   (1) Prove the three 'simple facts' from #4 above (with words Prove this!).

   (2) Show that the following sets are countable

      (a) All binary codes (arbitrary finite combinations of zeroes and ones).

      (b) All possible texts that can be typed on the standard keyboard.

      (c) All polynomials $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ with integer coefficients $a_k \in \mathbb{Z}$.

      (d) The set of all periodic sequences of rational numbers (i.e. sequences $a_1, a_2, a_3, \ldots$ with $a_k \in \mathbb{Q}$ and $a_{k+d} = a_k$ for some $d \in \mathbb{N}$ and all $k \in \mathbb{N}$).

   (3) Any collection of disjoint intervals on $\mathbb{R}$

   (4) Non-intersecting disks on the plane

   (5*) Non-intersecting figure eights on the plane

   (6) Prove that if X is infinite and Y is countable then $|X \cup Y| = |X|$.

   (7) Prove that any collection of non-intersecting intervals on the real line is countable.

   (8*) A collection of non-intersecting figure 8's of arbitrary size (nesting is allowed) is drawn on the plane. Prove that this collection is countable.