BMC Advanced
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Comparing Infinities II

1. First examples of uncountable sets.

   (a) Denote by $\mathcal{S}$ the set of all binary sequences, i.e. of infinite sequences $a_1, a_2, a_3, \ldots$, where each a_k is equal to either 0 or 1.

   (b) The set $\mathcal{S}$ of sequences can be identified with the set of all functions from $\mathbb{N}$ to $\{0,1\}$: a function $f : \mathbb{N} \to \{0,1\}$ is represented by the sequence $f(1), f(2), f(3), \ldots$.

   (c) $\mathcal{S}$ can also be identified with the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$: a sequence (a_n) corresponds to the subset $\{n \in \mathbb{N} \,|\, a_n = 1\}$.

   (d) Theorem. The set $\mathcal{S}$ of all binary sequences is not countable.

   We proved it using Cantor's *diagonal method.*

   (e) Corollary. $|\mathbb{N}| < |\mathcal{S}| = |\mathcal{P}(\mathbb{N})|$.

2. Two general results about cardinality

   (a) **Cantor's Theorem.** Denote by $\mathcal{P}(X)$ the set of all subsets of a set X. Then for every set X, we have $|X| < |\mathcal{P}(X)|$, i.e. the set of subsets has a greater cardinality than X.

   Sketch of a proof. Assume that there exists a surjection $f : X \to \mathcal{P}(X)$. Consider the subset $Y := \{x \in X \,|\, x \notin f(x)\}$. Since f is onto, there is $y \in X$ such that $f(y) = Y$. By definition of Y, this implies that $y \notin Y$. But then, again by definition of Y, this means that $y \in Y$. Contradiction.

   (b) **Cantor-Bernstein Theorem.** If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.

3. Definition. We say that a set X has *cardinality of the continuum* if $|X| = |2^{\mathbb{N}}| = \mathcal{P}(\mathbb{N})$.

4. Examples. Each of the following sets has cardinality of the continuum.

   - Open unit interval $(0,1) = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$.
   - The set $\mathbb{R}$ of real numbers.
   - The set of all irrational numbers, i.e. $\mathbb{R} - \mathbb{Q}$.
   - The set of all transcendental numbers (recall that they are real numbers which are not roots of any polynomial with rational coefficients).
   - The open square $(0,1)^2 = (0,1) \times (0,1) = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1, 0 < y < 1\}$.
   - The plane $\mathbb{R}^2$.

5. Exercises Show that each of the following sets has the cardinality of the continuum.

   (a) Closed interval $[0,1]$.

   (b) The union of two closed intervals, $[0,1] \cap [2,3]$.

   (c) The closed square $[0,1]^2 = [0,1] \times [0,1]$.

   (d) The interval $(0,1)$ from which a countable subset removed.

   (e) Any subset of the plane containing an arc of some circle.

(f) The set of all decimal sequences, i.e. $\{a_1, a_2, a_3, \dots \mid a_i \in \{0, 1, 2, \dots, 9\}\}$.

(g) The set of all sequences of natural numbers, i.e. $a_1, a_2, a_3, \dots$, where $a_k \in \mathbb{N}\}$.

(h) The set of all sequences of real numbers.

(i) The set of all straight lines on the plane.