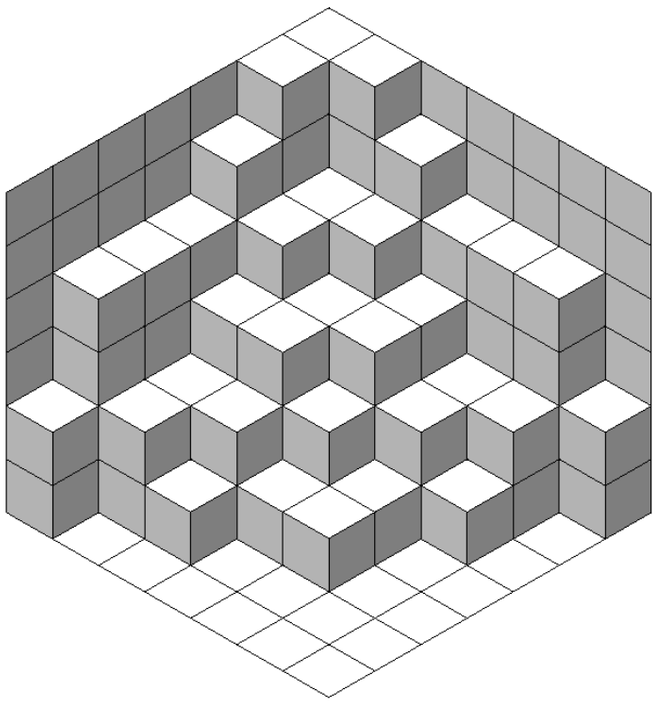
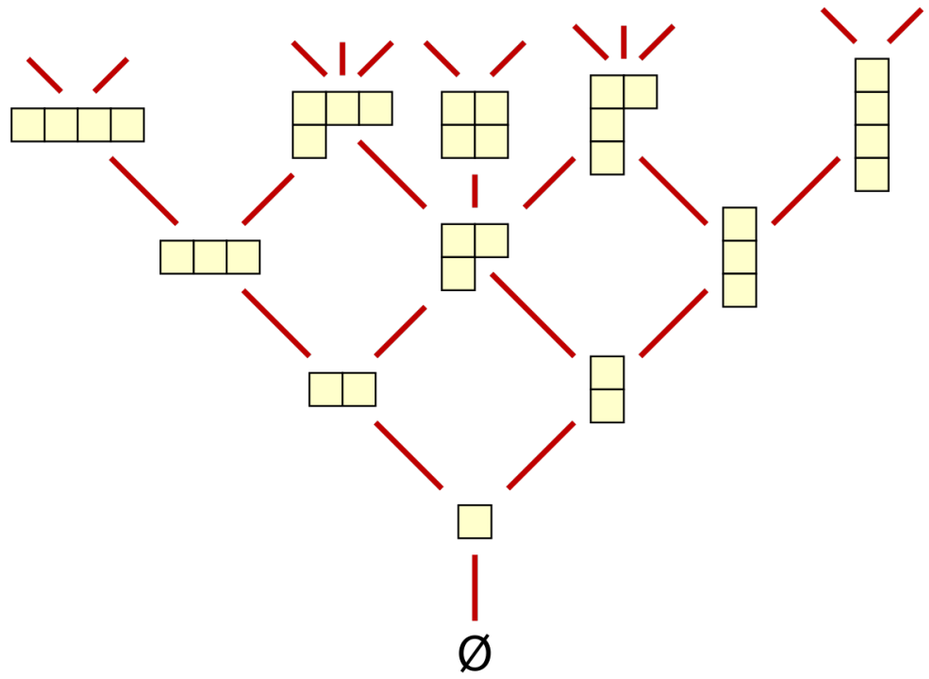
Berkeley Math Circle



Partitions

Peter Koroteev
UC Berkeley

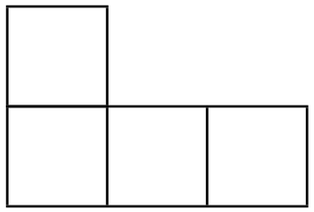


Partitions

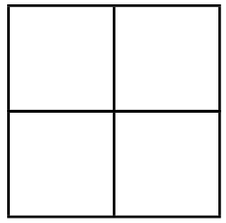
There are several ways to decompose an integer into sums of smaller integers

4=1+1+1+1

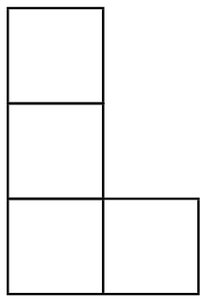
4=2+1+1



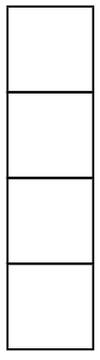
4=2+2



4=3+1



4=4+0

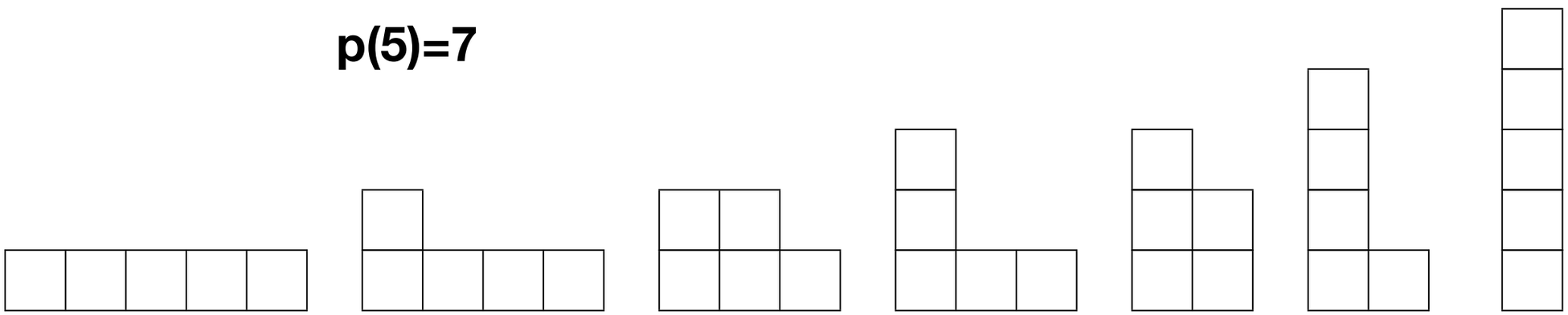


Young diagrams

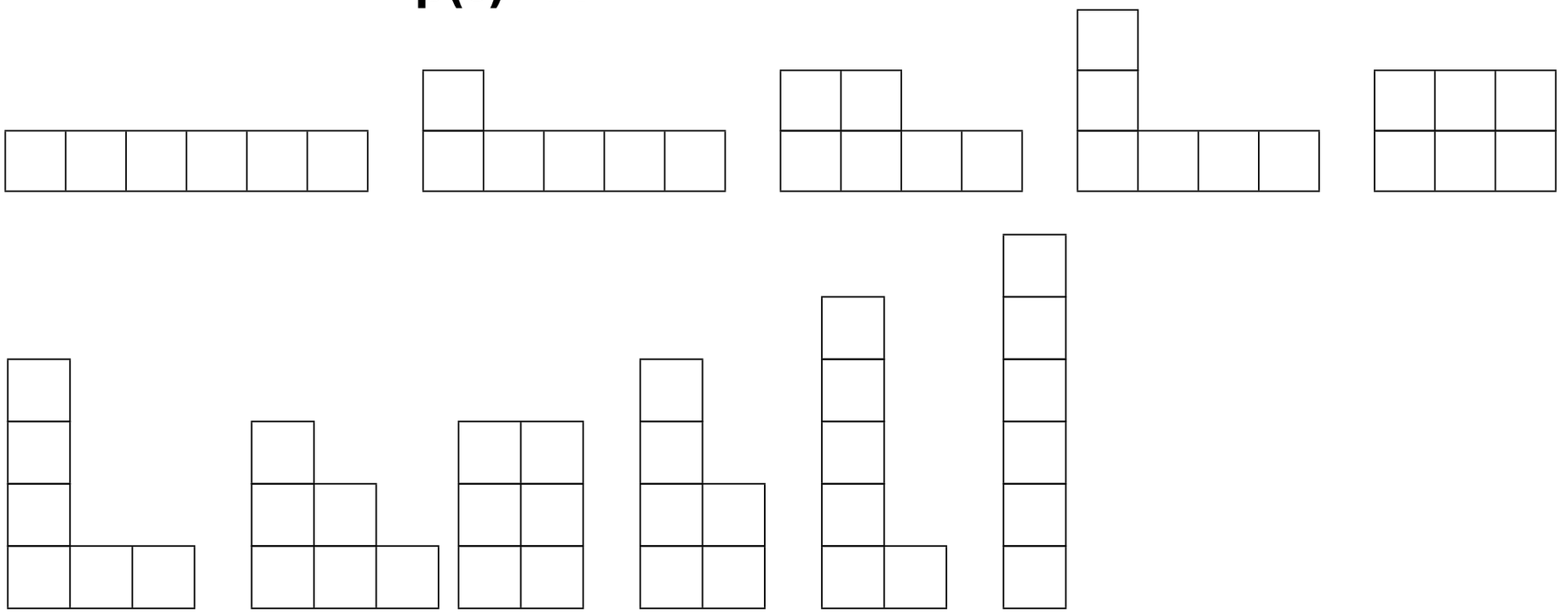
Partitions

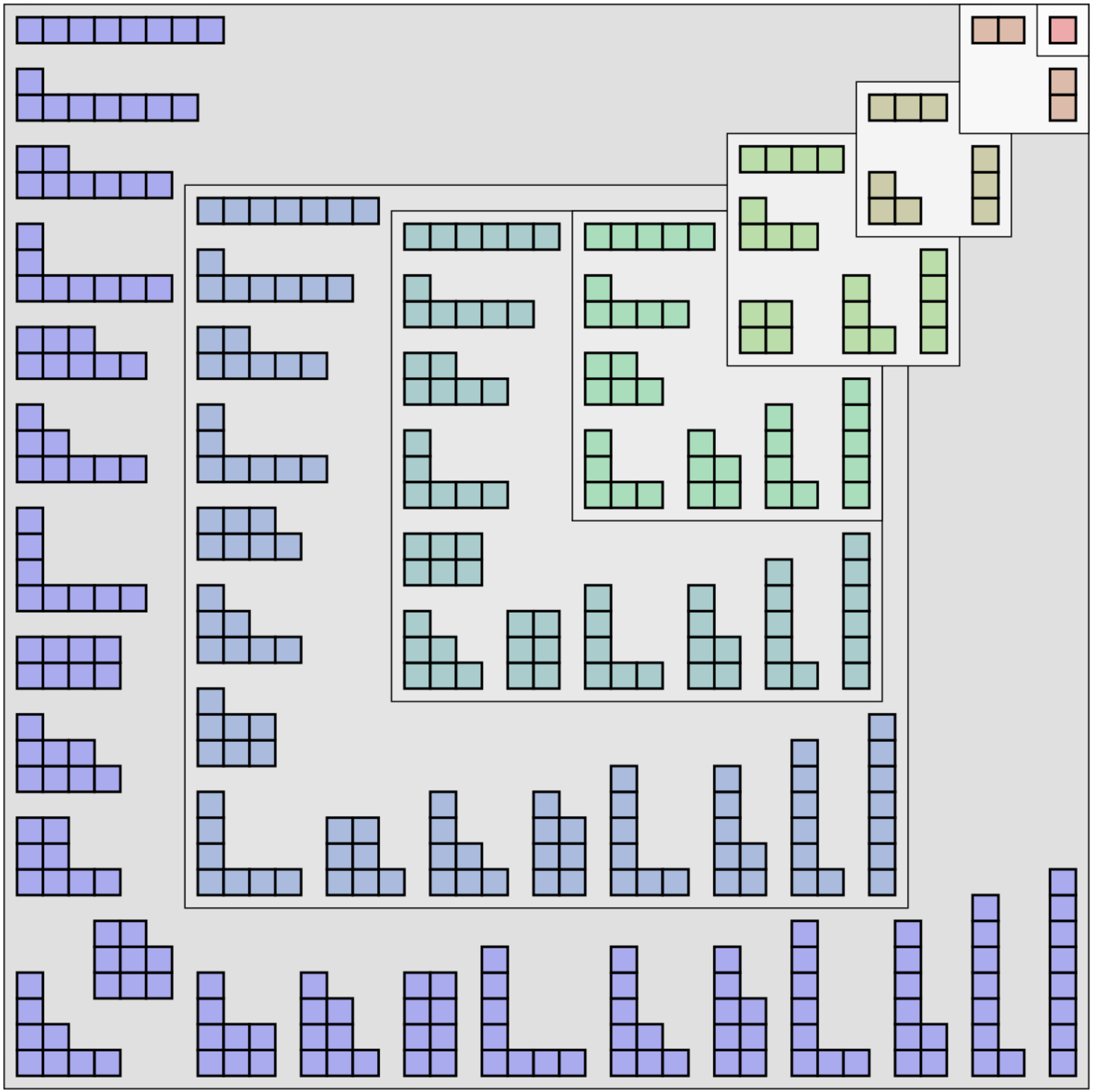
Problem: Find all partitions of numbers 1,2,3,4,5,6,7,8,9 together with their Young diagrams

n=5 **p(5)=7**



n=6 **p(6)=11**





Partitions

n=7 **p(7)=15**

{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {4, 1, 1, 1}, {3, 3, 1}, {3, 2, 2}, {3, 2, 1, 1},
{3, 1, 1, 1, 1}, {2, 2, 2, 1}, {2, 2, 1, 1, 1}, {2, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}

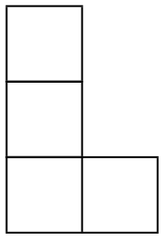
n=8 **p(8)=22**

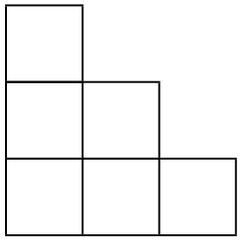
{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {5, 1, 1, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {4, 2, 1, 1},
{4, 1, 1, 1, 1}, {3, 3, 2}, {3, 3, 1, 1}, {3, 2, 2, 1}, {3, 2, 1, 1, 1}, {3, 1, 1, 1, 1, 1}, {2, 2, 2, 2},
{2, 2, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}

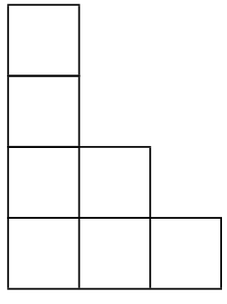
n=9 **p(9)=30**

{9}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1},
{5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1},
{3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1},
{3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1},
{1, 1, 1, 1, 1, 1, 1, 1, 1}

Odd & Distinct Parts







Problem (a): Count the number of partitions with ***odd parts*** from the previous examples

Problem (b): Count the number of partitions with ***distinct parts*** from the previous examples

n=7 **p(7)=15**

{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {4, 1, 1, 1}, {3, 3, 1}, {3, 2, 2}, {3, 2, 1, 1}, {3, 1, 1, 1, 1}, {2, 2, 2, 1}, {2, 2, 1, 1, 1}, {2, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}

n=8 **p(8)=22**

{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {5, 1, 1, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {4, 2, 1, 1}, {4, 1, 1, 1, 1}, {3, 3, 2}, {3, 3, 1, 1}, {3, 2, 2, 1}, {3, 2, 1, 1, 1}, {3, 1, 1, 1, 1, 1}, {2, 2, 2, 2}, {2, 2, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}

n=9 **p(9)=30**

{9}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}

Odd & Distinct Partitions

Find **odd and distinct** partitions for n = 1,2,…,11

n=9, p(9)=30

$$\{9\},\{8,1\},\{7,2\},\{7,1,1\},\{6,3\},\{6,2,1\},\{6,1,1,1\},\{5,4\},\{5,3,1\},\{5,2,2\},\{5,2,1,1\},$$
$$\{5,1,1,1,1\},\{4,4,1\},\{4,3,2\},\{4,3,1,1\},\{4,2,2,1\},\{4,2,1,1,1\},\{4,1,1,1,1,1\},\{3,3,3\},$$
$$\{3,3,2,1\},\{3,3,1,1,1\},\{3,2,2,2\},\{3,2,2,1,1\},\{3,2,1,1,1,1\},\{3,1,1,1,1,1,1\},\{2,2,2,2,1\}$$
$$\{2,2,2,1,1,1\},\{2,2,1,1,1,1,1\},\{2,1,1,1,1,1,1,1\},\{1,1,1,1,1,1,1,1,1\}$$

n=10, p(10)=42

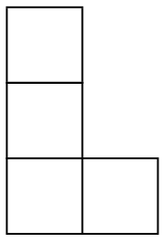
$$\{\{10\},\{9,1\},\{8,2\},\{8,1,1\},\{7,3\},\{7,2,1\},\{7,1,1,1\},\{6,4\},\{6,3,1\},\{6,2,2\},\{6,2,1,1\},$$
$$\{6,1,1,1,1\},\{5,5\},\{5,4,1\},\{5,3,2\},\{5,3,1,1\},\{5,2,2,1\},\{5,2,1,1,1\},\{5,1,1,1,1,1\},$$
$$\{4,4,2\},\{4,4,1,1\},\{4,3,3\},\{4,3,2,1\},\{4,3,1,1,1\},\{4,2,2,2\},\{4,2,2,1,1\},\{4,2,1,1,1,1\},$$
$$\{4,1,1,1,1,1,1\},\{3,3,3,1\},\{3,3,2,2\},\{3,3,2,1,1\},\{3,3,1,1,1,1\},\{3,2,2,2,1\},\{3,2,2,1,1,1\}$$
$$\{3,2,1,1,1,1,1\},\{3,1,1,1,1,1,1,1\},\{2,2,2,2,2\},\{2,2,2,2,1,1\},\{2,2,2,1,1,1,1\},$$
$$\{2,2,1,1,1,1,1,1\},\{2,1,1,1,1,1,1,1,1\},\{1,1,1,1,1,1,1,1,1,1\}\}$$

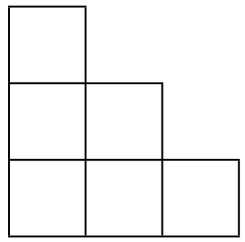
n=11, p(11)=56

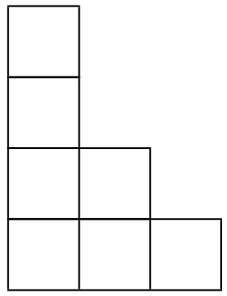
$$\{\{11\},\{10,1\},\{9,2\},\{9,1,1\},\{8,3\},\{8,2,1\},\{8,1,1,1\},\{7,4\},\{7,3,1\},\{7,2,2\},\{7,2,1,1\},$$
$$\{7,1,1,1,1\},\{6,5\},\{6,4,1\},\{6,3,2\},\{6,3,1,1\},\{6,2,2,1\},\{6,2,1,1,1\},\{6,1,1,1,1,1\},\{5,5,1\},$$
$$\{5,4,2\},\{5,4,1,1\},\{5,3,3\},\{5,3,2,1\},\{5,3,1,1,1\},\{5,2,2,2\},\{5,2,2,1,1\},\{5,2,1,1,1,1\},$$
$$\{5,1,1,1,1,1,1\},\{4,4,3\},\{4,4,2,1\},\{4,4,1,1,1\},\{4,3,3,1\},\{4,3,2,2\},\{4,3,2,1,1\},\{4,3,1,1,1,1\},$$
$$\{4,2,2,2,1\},\{4,2,2,1,1,1\},\{4,2,1,1,1,1,1\},\{4,1,1,1,1,1,1,1\},\{3,3,3,2\},\{3,3,3,1,1\},\{3,3,2,2,1\}$$
$$\{3,3,2,1,1,1\},\{3,3,1,1,1,1,1\},\{3,2,2,2,2\},\{3,2,2,2,1,1\},\{3,2,2,1,1,1,1\},\{3,2,1,1,1,1,1,1\},$$
$$\{3,1,1,1,1,1,1,1,1\},\{2,2,2,2,2,1\},\{2,2,2,2,1,1,1\},\{2,2,2,1,1,1,1,1\},\{2,2,1,1,1,1,1,1,1\},$$
$$\{2,1,1,1,1,1,1,1,1,1\},\{1,1,1,1,1,1,1,1,1,1,1\}\}$$

Odd vs. Distinct

n	1	2	3	4	5	6	7	8	9
p(n)	1	2	3	5	7	11	15	22	30
# odd	1	1	2	2	3	4	5	6	8
# dist.	1	1	2	2	3	4	5	6	8







Problem: Why is the number of these partitions is the same for every n?

odd	distinct
5	5
3,1,1	4,1
1,1,1,1,1	3,2

odd	distinct
5,1	6
3,3	5,1
3,1,1,1	4,2
1,1,1,1,1,1	3,2,1

odd	distinct
7	7
5,1,1	6,1
3,3,1	5,2
3,1,1,1,1	4,3
1,1,1,1,1,1,1	4,2,1

odd	distinct
7,1	8
5,3	7,1
5,1,1,1	6,2
3,3,1,1	5,3
3,1,1,1,1,1	5,2,1
1,1,1,1,1,1,1,1	4,3,1

odd	distinct
9	9
7,1,1	8,1
5,1,1,1	7,2
5,3,1	6,3
3,3,3	6,2,1
3,3,1,1,1	5,4
3,1,1,1,1,1,1	5,3,1
1,1,1,1,1,1,1,1,1	4,3,2

Matching

From Distinct to Odd.

$$2 \to 1,1$$
$$4 \to 1,1,1,1$$
$$6 \to 3,3$$
$$8 \to 1,1,1,1,1,1,1,1$$

$$8 \to 1,1,1,1,1,1,1,1$$
$$7,1 \to 7,1$$
$$6,2 \to 3,3,1,1$$
$$5,3 \to 5,3$$
$$5,2,1 \to 5,1,1,1$$
$$4,3,1 \to 3,1,1,1,1,1$$

From Odd to Distinct.

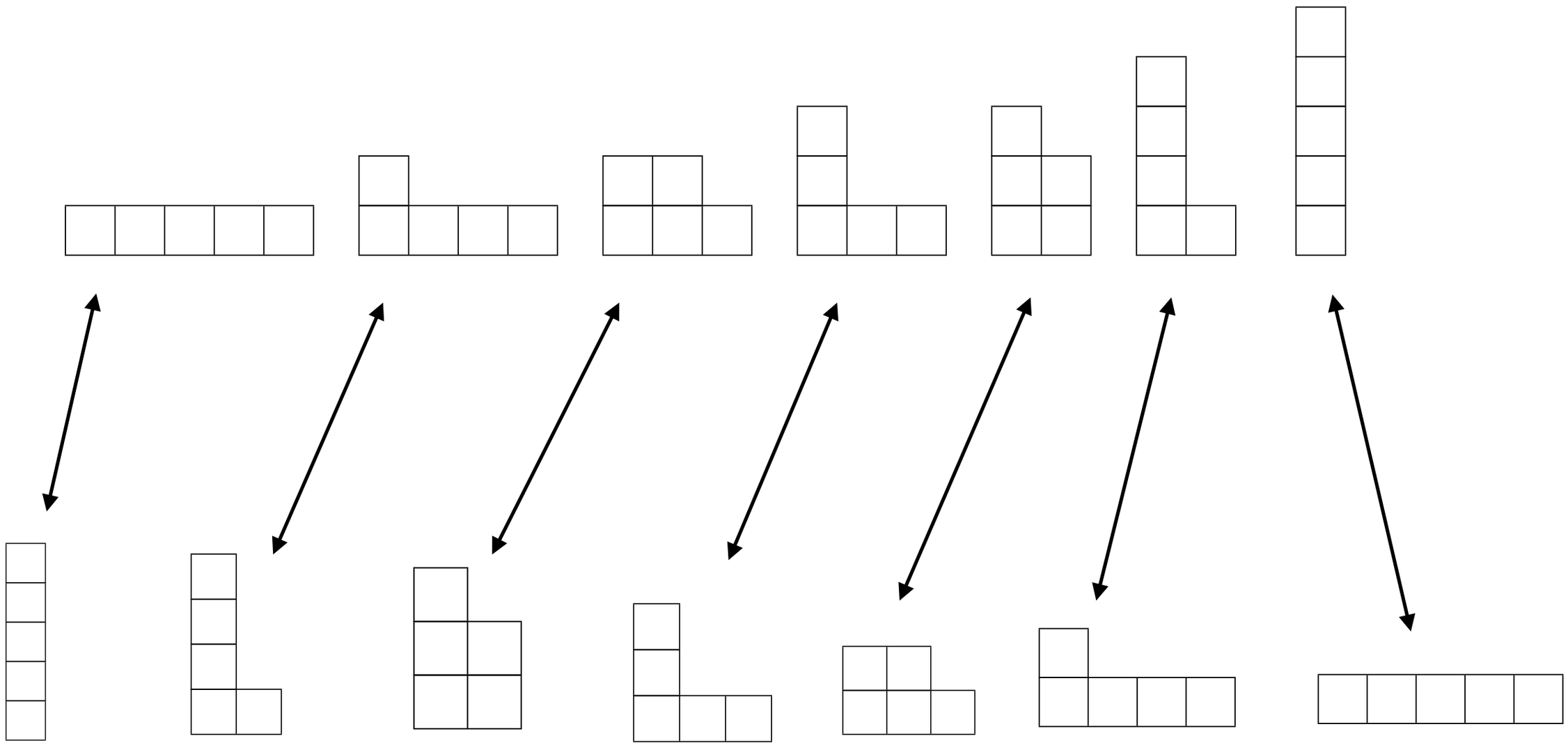
$$1 \to 1$$
$$1,1 \to 2$$
$$1,1,1 \to 2,1$$
$$1,1,1,1 \to 4$$
$$1,1,1,1,1 \to 4,1$$
$$1,1,1,1,1,1 \to 4,2$$
$$1,1,1,1,1,1,1 \to 4,2,1$$
$$1,1,1,1,1,1,1,1 \to 8$$
$$1,1,1,1,1,1,1,1,1 \to 8,1$$

Binary presentation

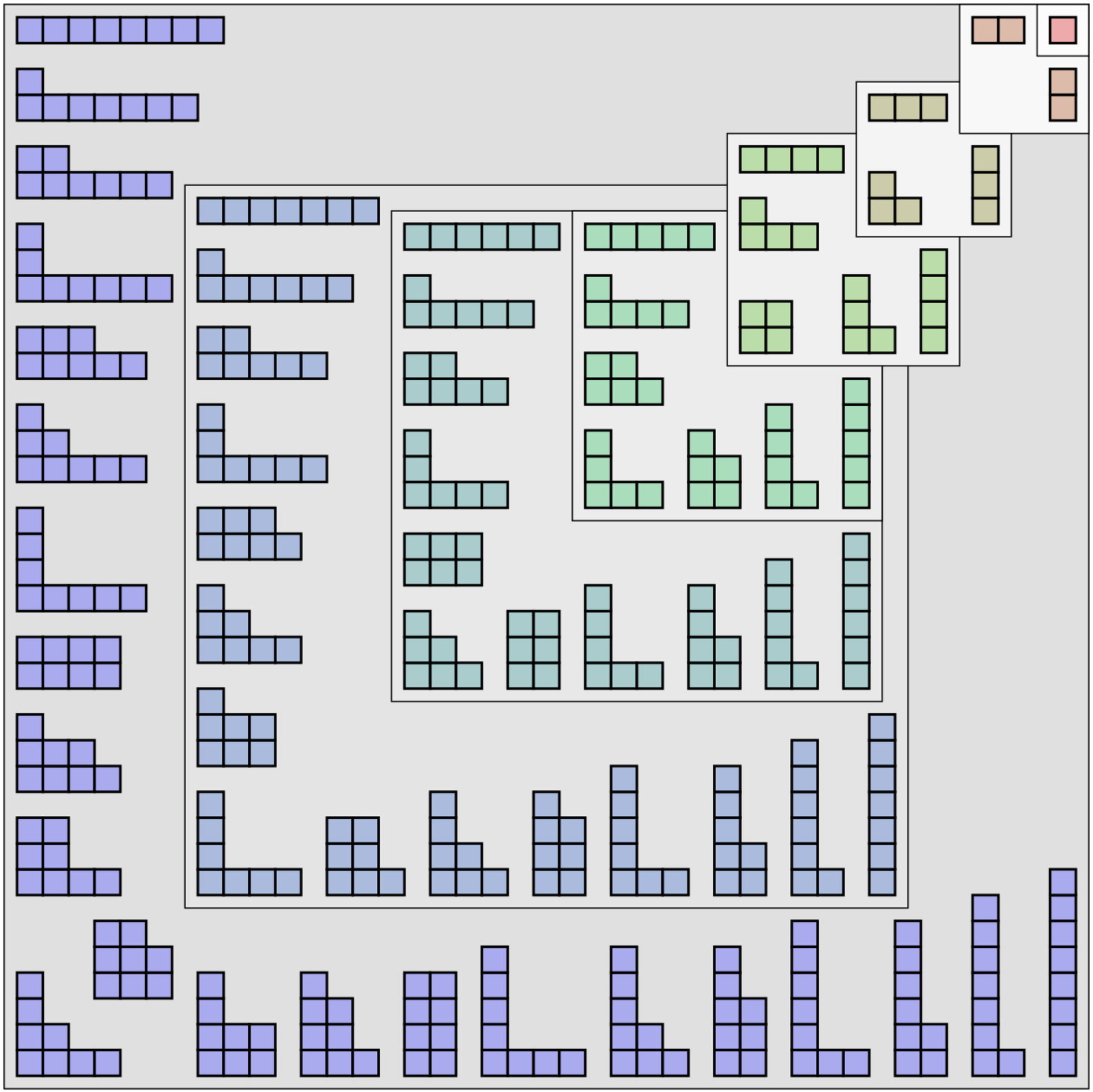
$$3 = 2^1 + 2^0\,, \qquad 5 = 2^2 + 2^0\,, \qquad 8 = 2^3$$

Conjugated Partitions

Flip Young diagrams over the diagonal



Problem: how many self-conjugated (symmetric) partitions are there?



p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

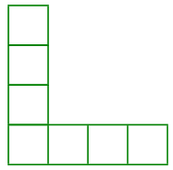
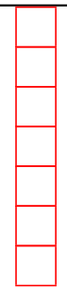
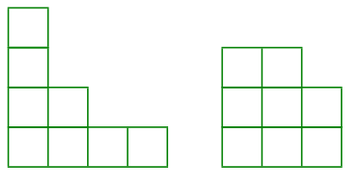
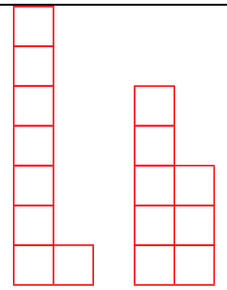
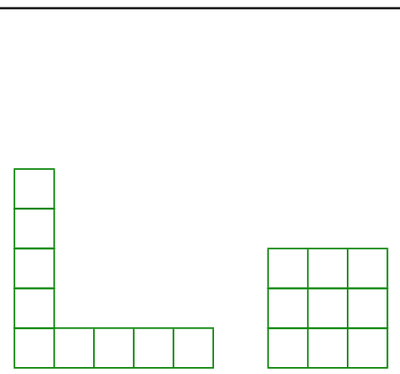
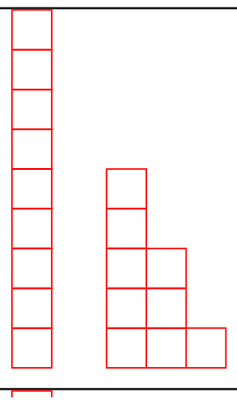
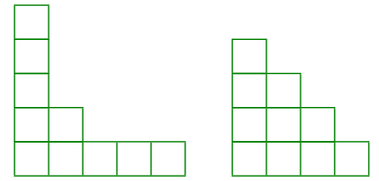
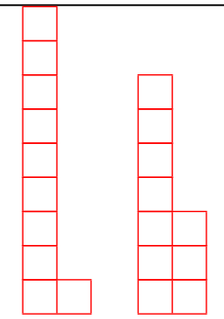
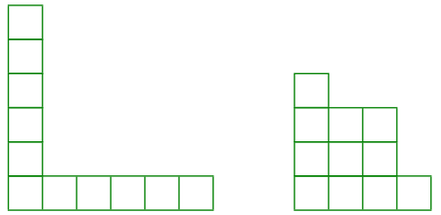
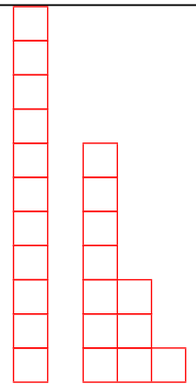
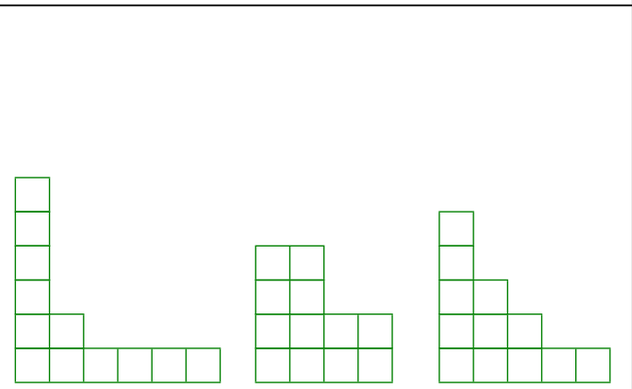
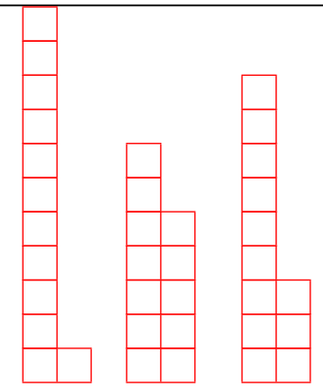
p(11)

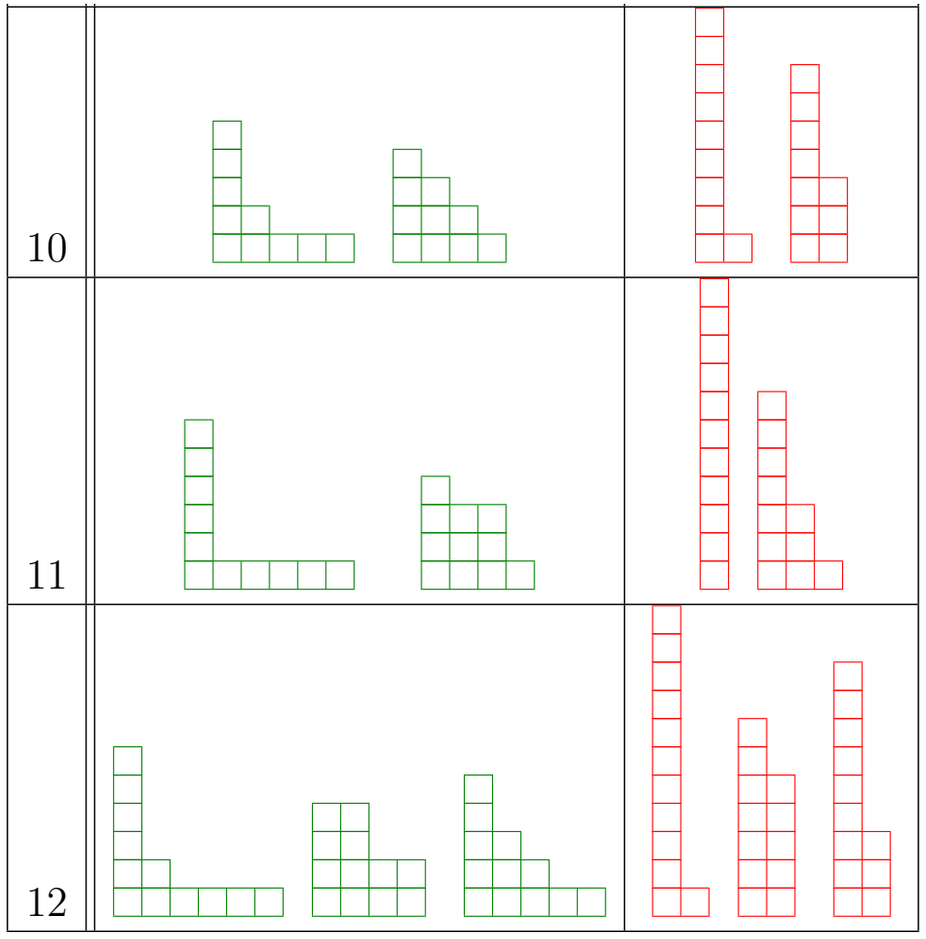
{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

Odd, Distinct, Symmetric

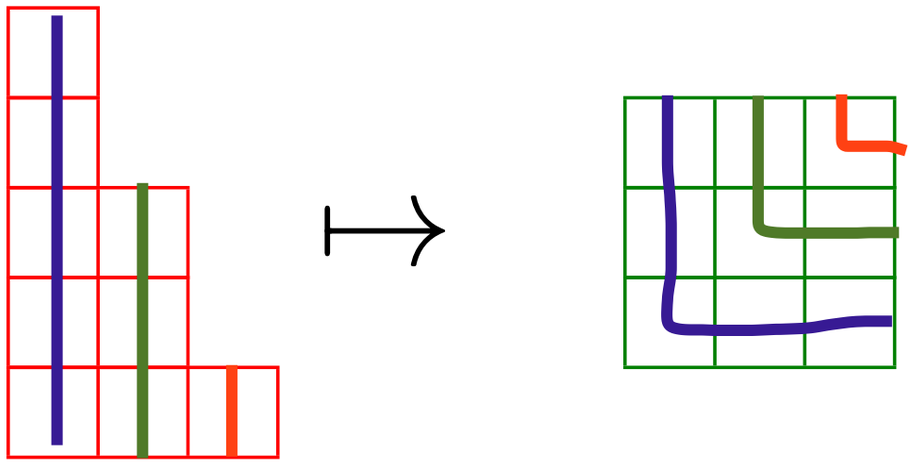
n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Problem: Show that Symmetric = Odd&Distinct

n	Symmetric	Odd&Distinct
7		
8		
9		
10		
11		
12		

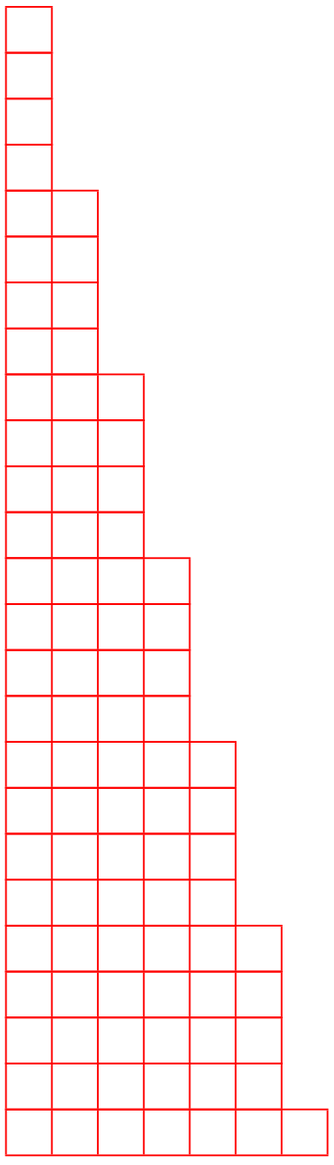

10
11
12

Matching

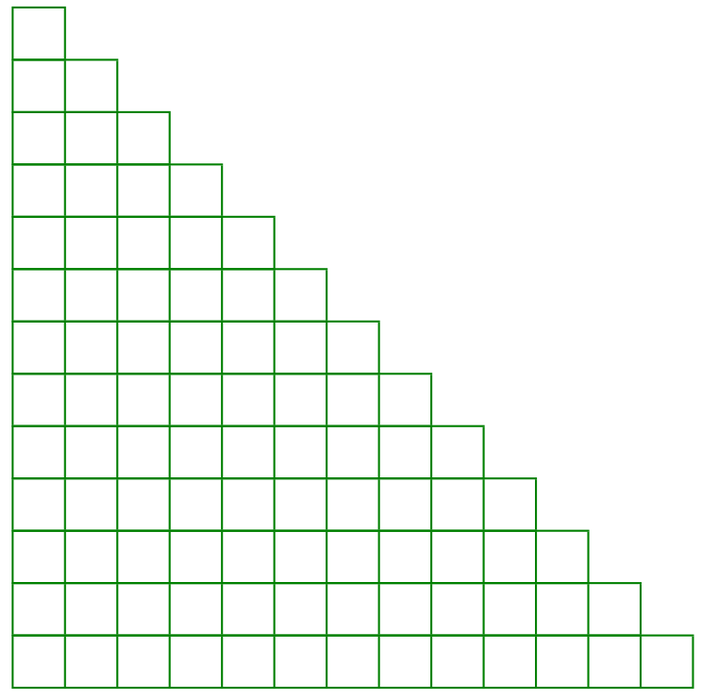


Odd&Distinct vs Symmetric

Triangular numbers

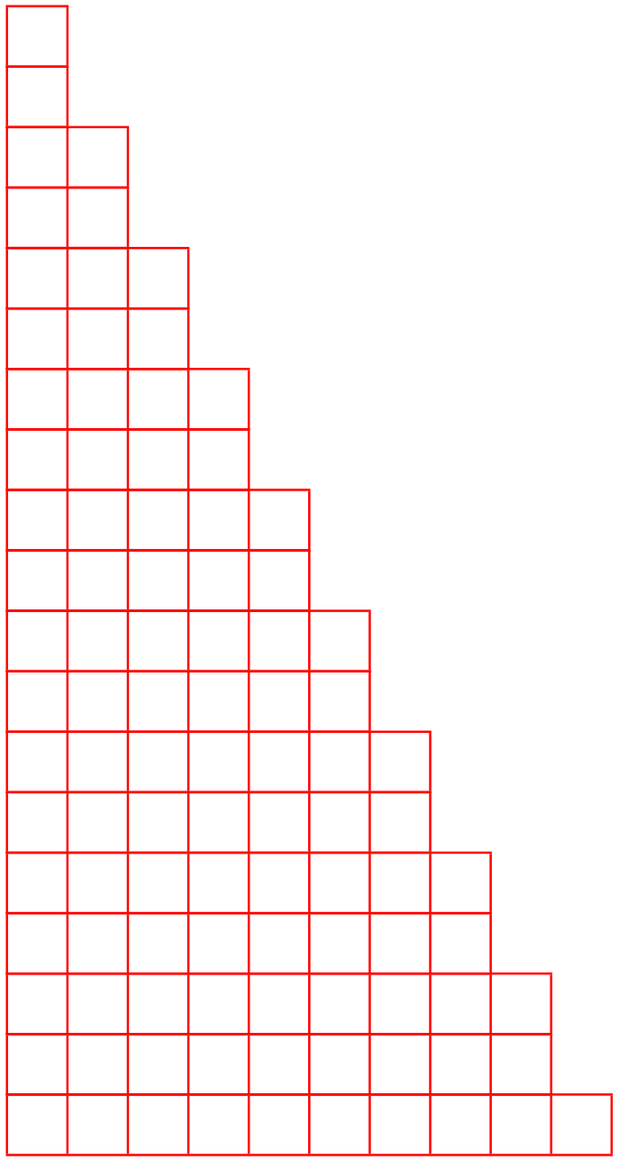


{25,21,17,13,9,5,1}

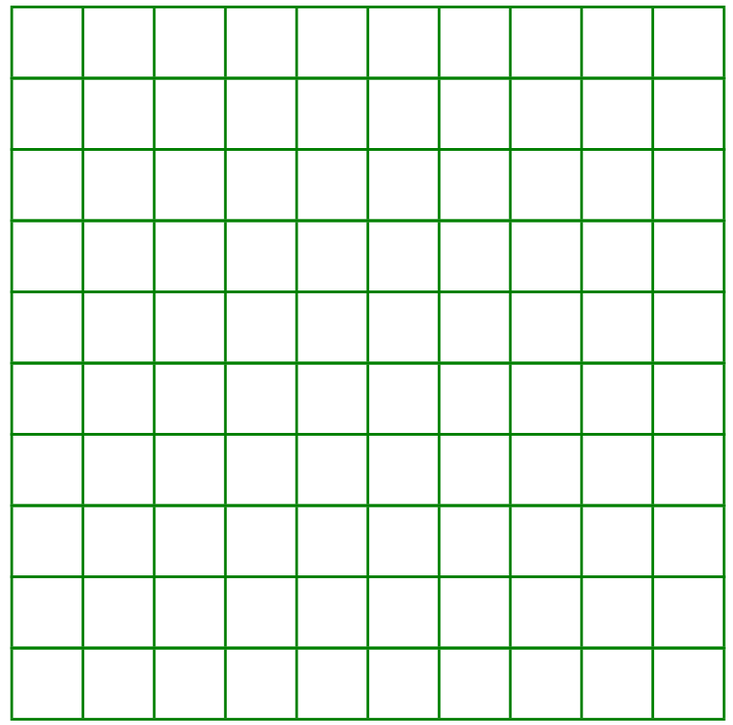


{13,12,11,10,9,8,7,6,5,4,3,2,1

Odd&Distinct vs Symmetric
Square numbers



$\{19, 17, 15, 13, 11, 9, 7, 5, 3, 1\}$



$\{10, 10, 10, 10, 10, 10, 10, 10, 10, 10\}$

Computing Sums

Compute the sum. How many terms are in the sum?

- $1 + 3 + 5 + \ldots + 117 + 119 = ?$

- $1 + 5 + 9 + 13 + 17 + 21 + 25 = ?$

- $1 + 5 + 9 + 13 + \ldots + 81 = ?$

Divisibility by 3

Consider partitions of n whose parts are not divisible by **3**

Compare those with partitions of n in which **each part** is not repeated **3 or more times**

Divisibility by 4

Consider partitions of n whose parts are not divisible by **4**

Compare those with partitions of n in which **each part** is not repeated **4 or more times**

Divisibility by n

Consider partitions of n whose parts are not divisible by **n**

Compare those with partitions of n in which **each part** is not repeated **n** **or more times**

Restricted Partitions

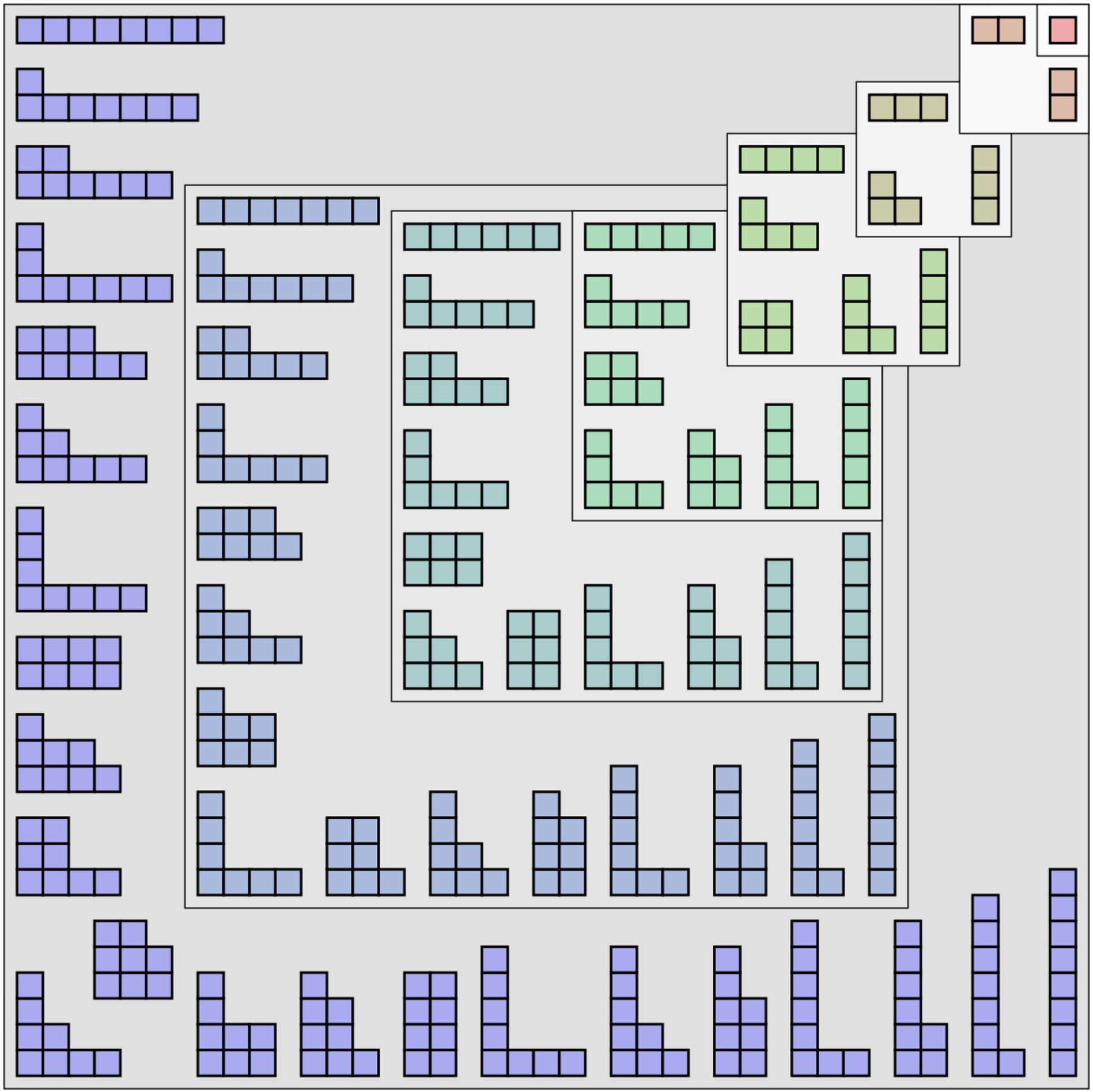
Restricted Partitions. Let us now look at integer partitions of n which have *exactly* 4 parts. From the list of partitions of 7

$$\{\{7\},\{6,1\},\{5,2\},\{5,1,1\},\{4,3\},\{4,2,1\},\{4,1,1,1\},\{3,3,1\},\{3,2,2\},$$
$$\{3,2,1,1\},\{3,1,1,1,1\},\{2,2,2,1\},\{2,2,1,1,1\},\{2,1,1,1,1,1\},\{1,1,1,1,1,1,1\}\}$$

Only these qualify

$$\{\{4,1,1,1\},\{3,2,1,1\},\{2,2,2,1\}\}$$

Make lists for $n = 4, 5, 6$ with all possible restricted parts, i.e. all partitions of 4 with 1, with 2, with 3 parts, etc., same for $n = 5$ and $n = 6$. Count each number of partitions, call it $p_k(n)$. Do you see any pattern?



	1	2	3	4	5	6	7	8
p(1)	1							
p(2)	1	1						
p(3)	1	1	1					
p(4)	1	2	1	1				
p(5)	1	2	2	1	1			
p(6)	1	3	3	2	1	1		
p(7)	1	3	4	3	2	1	1	
p(8)	1	4	5	5	3	2	1	1

p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(11)

{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

	1	2	3	4	5	6	7	8	9	10	11	12
p(1)	1											
p(2)	1	1										
p(3)	1	1	1									
p(4)	1	2	1	1								
p(5)	1	2	2	1	1							
p(6)	1	3	3	2	1	1						
p(7)	1	3	4	3	2	1	1					
p(8)	1	4	5	5	3	2	1	1				
p(9)	1	4	7	6	5	3	2	1	1			
p(10)	1	5	8	9	7	5	3	2	1	1		
p(11)												
p(12)												

p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(11)

{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}